Technical report #1

presented to

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of

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by

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Introduction

This technical report was made as a supplementary part of the communication between the author and his supervisor, Professor Katsuhisa Furuta of the Tokyo Institute of Technology, during his time as a Ph.D. student there.

Though the author call it a technical report, it is not always true that all of the material put in this report is new or novel. Since the author has not been very fussy in his choise of problems, the problems presented in here cover many thing and many of them are nothing more than interesting exercises.

A system with absolute functions

22^{th} April 1998

Figure 1 shows the phase plane of the system [Kha96] (Exercise 1.17(4))

$$\left.\begin{aligned} \dot{x_1} &= x_1 + x_2 - x_1\left(|x_1| + |x_2|\right), \\ \dot{x_2} &= -2x_1 + x_2 - x_2\left(|x_1| + |x_2|\right) + u, \end{aligned}\right\} \tag{1}$$

when $u = 0$.

Preliminary knowledge about the system

The simultaneous equations

$$\left.\begin{aligned} x_1 + x_2 - x_1\left(|x_1| + |x_2|\right) &= 0, \\ -2x_1 + x_2 - x_2\left(|x_1| + |x_2|\right) &= 0 \end{aligned}\right\} \tag{2}$$

are ill-formed, therefore cannot be solved for equilibrium points.

Phase plot of the system

Figure 1 and Figure 2 show the phase plane plot of Equation 1

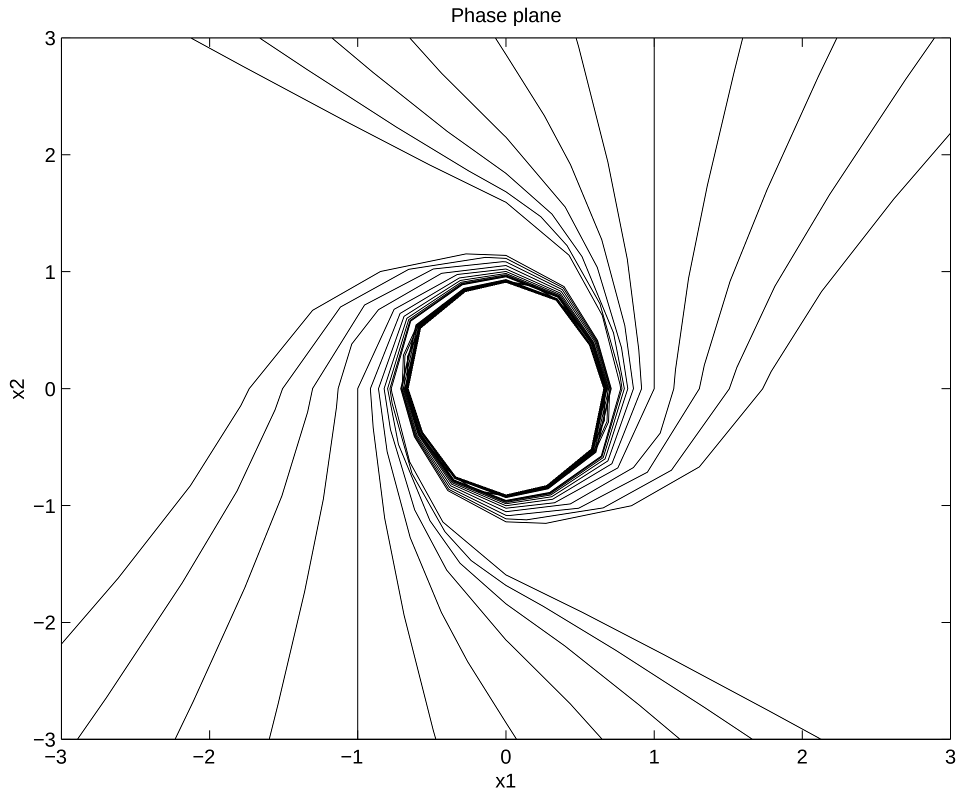


Figure 1: *Phase plane when there is no control.*

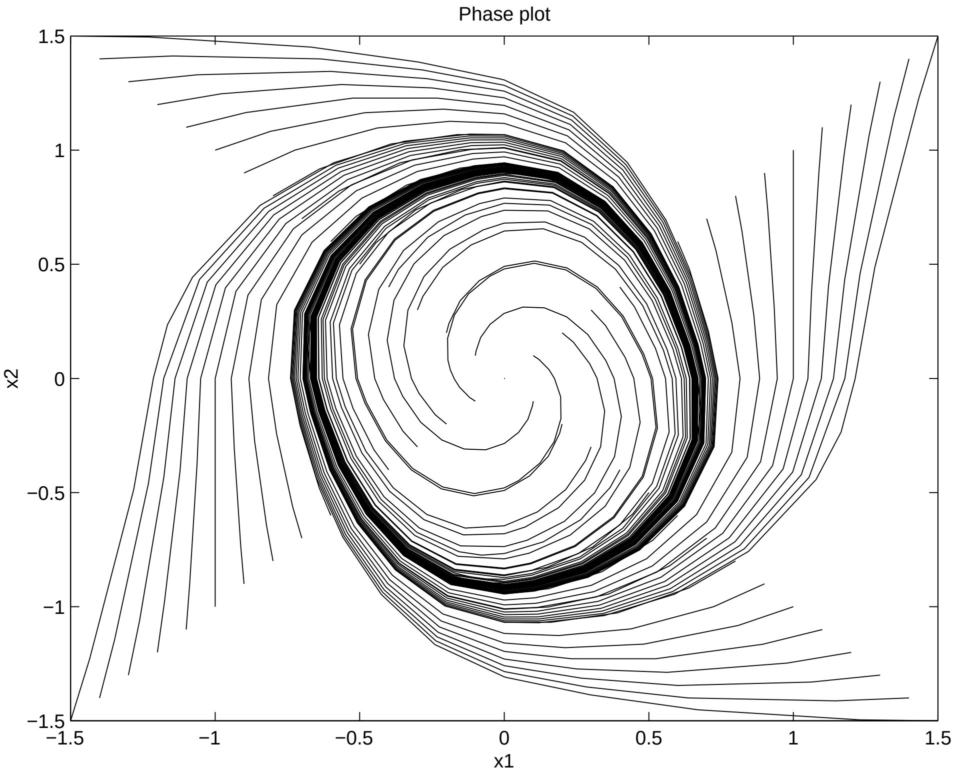


Figure 2: *Phase plane when there is no control.* $t_{CPU} = 1.24\,\mathrm{sec}$ *simulated on Hayate. Initial conditions* $(x_1, x_2) = (i, i)\,\mathrm{and}\,(i, -i),\ i = -1.5, -1.4, -1.3, \ldots, 1.5$

Feedback of Signum function

Figure 3 and Figure 4 show the phase plane when this system is subjected to an input

$$u = \mathrm{sgn}(x_1 + x_2). \tag{3}$$

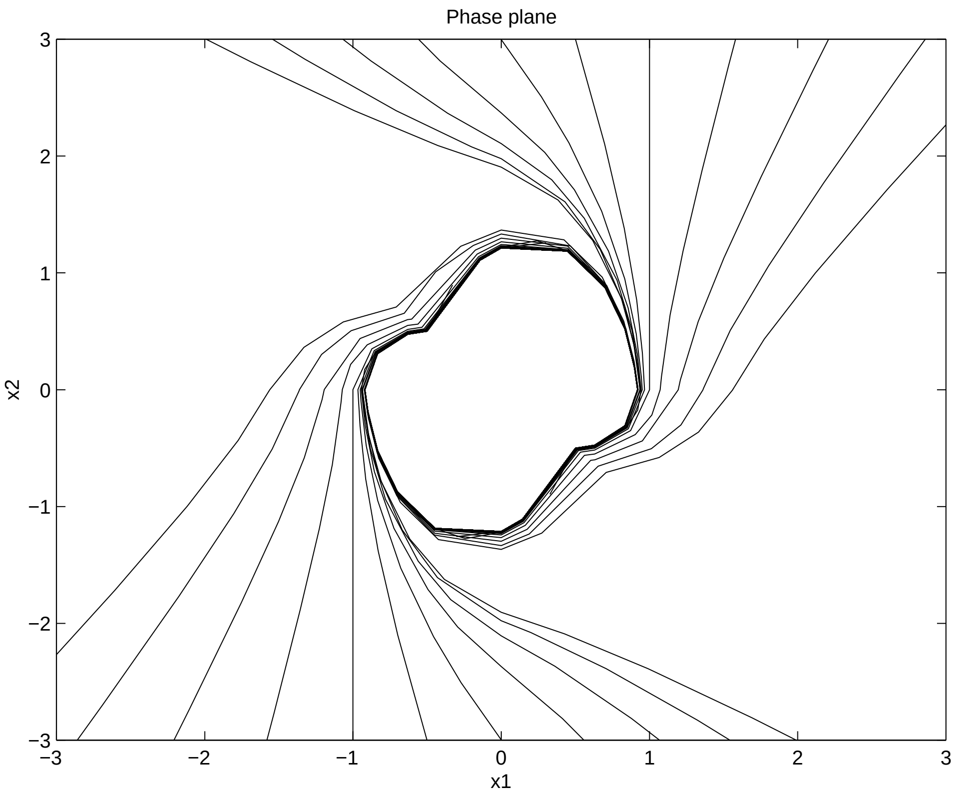


Figure 3: *Phase plane when the control input is $u = sgn(x_1 + x_2)$.*

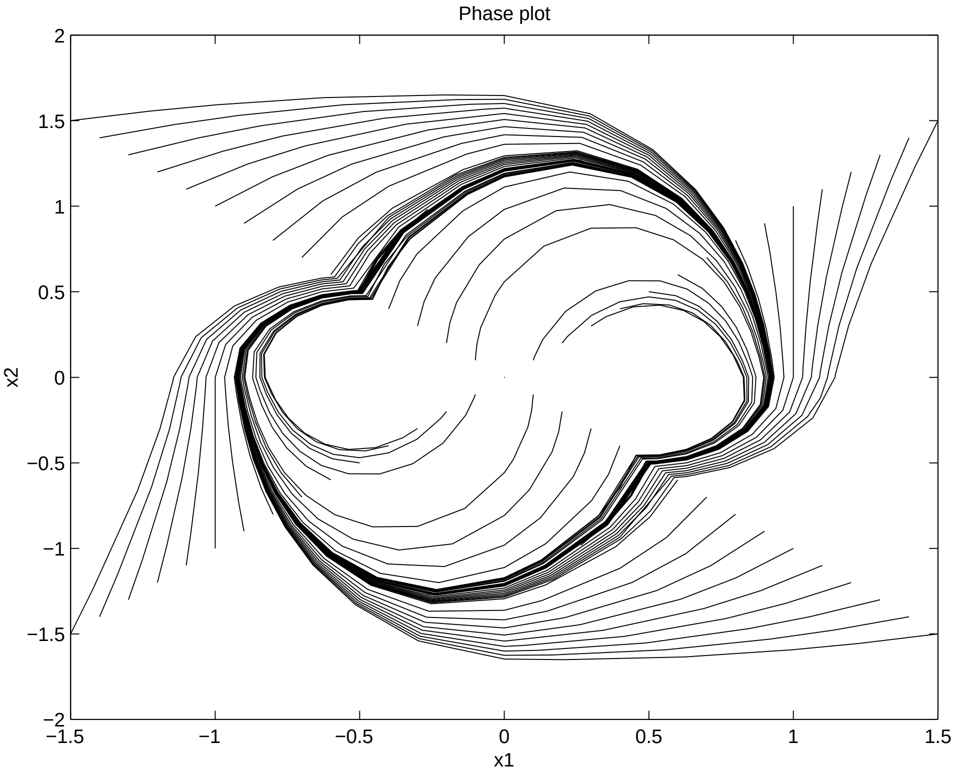


Figure 4: *Phase plane when the control input is* $u = sgn(x_1 + x_2)$. $t_{CPU} = 1.49\,\mathrm{sec}$ *simulated on Hayate. Initial conditions* $(x_1, x_2) = (i, i)$ and $(i, -i),\ i = -1.5, -1.4, -1.3, \ldots, 1.5$

Feedback of Signum function and states

Figure 5 and Figure 6 Show the phase plane when

$$u = -(x_1 + x_2) + \operatorname{sgn}(x_1 + x_2). \tag{4}$$

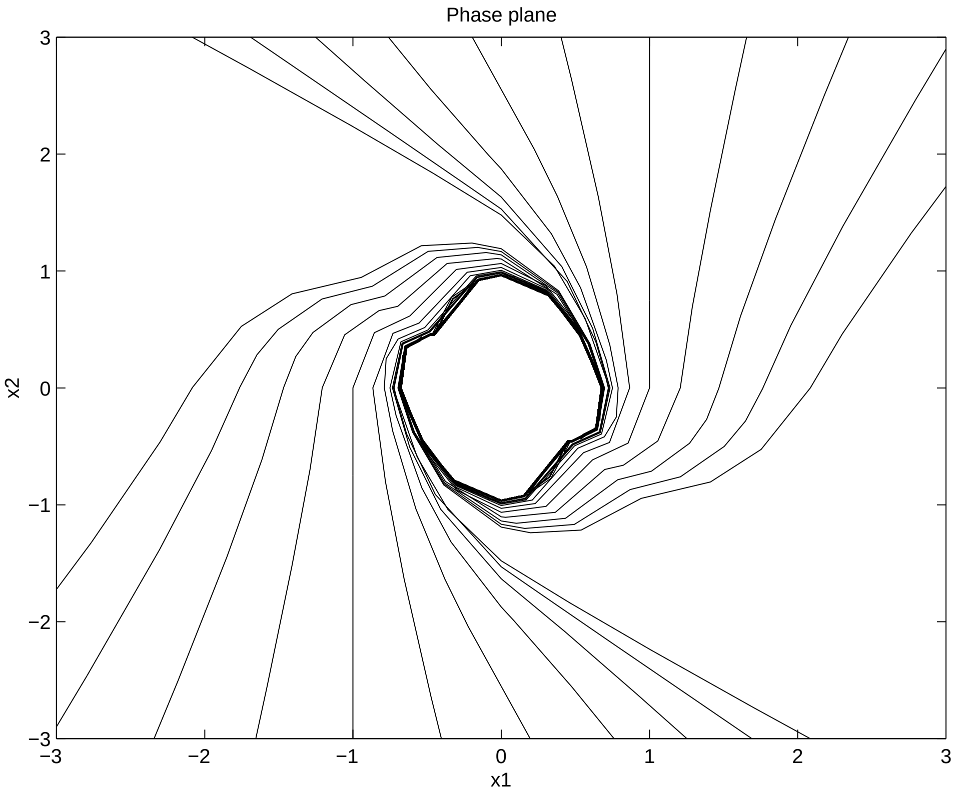


Figure 5: *Phase plane when the control is* $u = -(x_1 + x_2) + sgn(x_1 + x_2)$.

Discussion

When there was no control. Figure 1 and Figure 2 shows that the equilibrium point at the origin is an unstable focus and, also that there is a limit cycle circling around the origin. The limit cycle is slightly larger in the x_2 direction than in the x_1 direction. Every trajectory starting from an initial point other than the origin goes to and then stays on this limit cycle.

When the control input is $u = \operatorname{sgn}(x_1 + x_2)$. Figure 3 and Figure 4 shows the discontinuity on the surface of the hyperplane. ($s = x_1 + x_2 = 0$) All trajectories still converge to the limitcycle, though the limit cycle is also distorted where it intersects the hyperplane. There is no node. Figure 4 shows that every point which is on the hyperplane and inside the limit cycle acts as an unstable node.

When the control input is $u = -(x_1 + x_2) + \operatorname{sgn}(x_1 + x_2)$. Figure 5 and Figure 6 also shows the discontinuity on the hyperplane. Similar to Figure 4, Figure 6 shows that points on the hyperplane

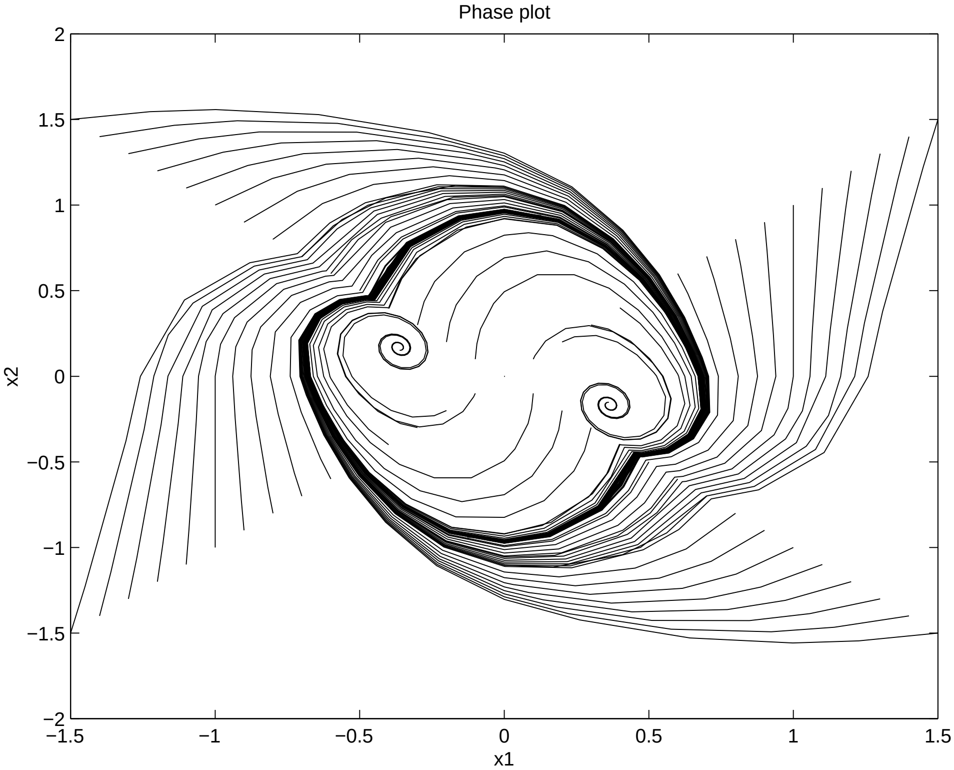


Figure 6: *Phase plane when the control is* $u = -(x_1 + x_2) + sgn(x_1 + x_2)$. $t_{CPU} = 1.53\,\mathrm{sec}$ *simulated on Hayate. Initial conditions* $(x_1, x_2) = (i, i)$ and $(i, -i),\ i = -1.5, -1.4, -1.3, \ldots, 1.5$

inside the limit cycle acts as an unstable node. In addition to Figure 4, Figure 6 has got two stable nodes.

Values of ε	nature of the origin
$-\infty < \varepsilon \leq -2$	stable node
$-2 < \varepsilon < 0$	stable focus
$\varepsilon = 0$	centre
$0 < \varepsilon < 2$	unstable focus
$2 \leq \varepsilon < \infty$	unstable node

Table 1: *Characteristics of trajectories of a perturbed system with periodicity as depending on* ε

Perturbed system with a periodic orbit

Next consider the system [Kha96] (Exercise 7.1(4))

$$\dot{x_1} = x_2 \tag{5}$$
$$\dot{x_2} = -x_1 + \varepsilon x_2 \left(1 - x_1^2 - x_2^2\right) + u, \quad \varepsilon > 0. \tag{6}$$

Some knowledge about the system

By solving the simultaneous equations

$$\begin{aligned} x_2 &= 0 \\ -x_1 + \varepsilon x_2 \left(1 - x_1^2 - x_2^2\right) &= 0 \end{aligned} \tag{7}$$

the equilibrium point is at $(0, 0)$.

The Jacobian matrix at this equilibrium point is

$$A = \left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{array} \right]\Bigg|_{x=0} = \left[\begin{array}{cc} 0 & 1 \\ -1 - 2\varepsilon xy & -2\varepsilon y^2 + \varepsilon(1 - x^2 - y^2) \end{array} \right]\Bigg|_{x=0} = \left[\begin{array}{cc} 0 & 1 \\ -1 & \varepsilon \end{array} \right].$$

The Eigenvalues are then $\frac{\varepsilon}{2} \pm \frac{1}{2}\sqrt{\varepsilon^2 - 4}$. Thus the nature of the equilibrium point at the origin depends upon the value of ε. The characteristic of the trajectories around the equilibrium point is shown graphically in Figure 7 and as an intrepretation of Figure 7 in Table 1.

Phase planes of various values of ε

Draw the phase plane when $u = 0$ of this system as shown in Figure 8 ($\varepsilon = 1 \times 10^{-7}$), Figure 9 ($\varepsilon = 1 \times 10^{-5}$), Figure 10 ($\varepsilon = 0.001$), Figure 11 ($\varepsilon = 0.01$), Figure 12 ($\varepsilon = 0.1$), Figure 13 ($\varepsilon = 1$) and Figure 14 ($\varepsilon = 5$). The initial conditions used were either

$$(x_1, x_2) \in \{5, -5\} \times \{i \in \{5 \leq \mathbb{I} \leq 5\}\}, or$$

or

$$(x_1, x_2) = (i, i), (i, -i), \quad i = -1, -0.8, -0.6, \ldots, 1.$$

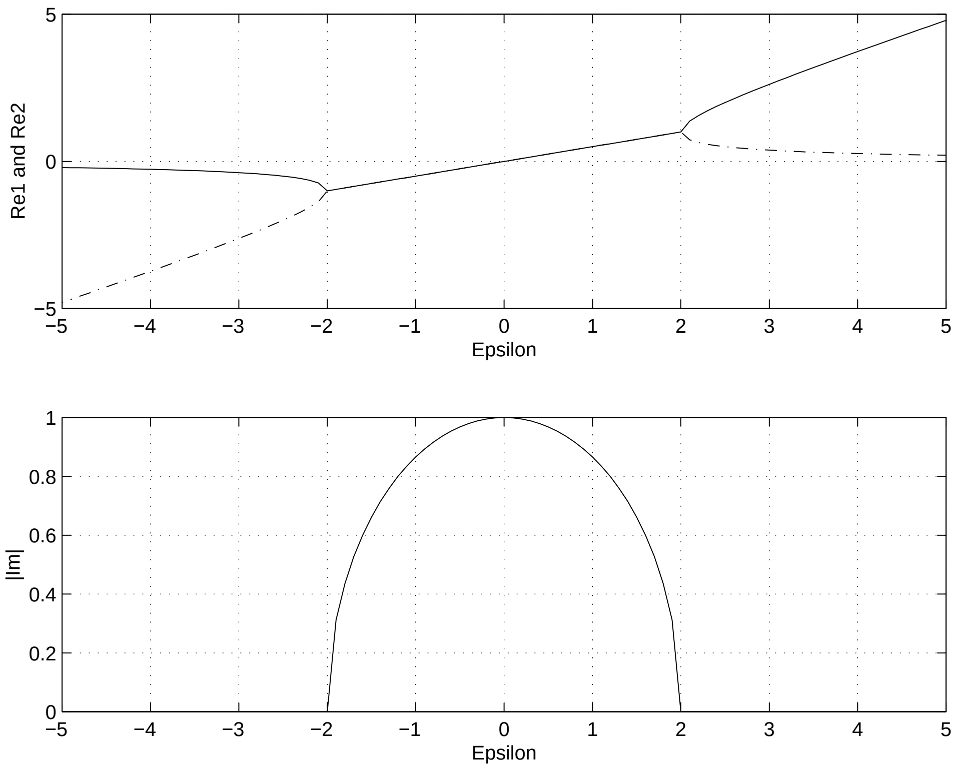


Figure 7: *Eigenvalues as a function of* ε *of a perturbed system with periodicity*

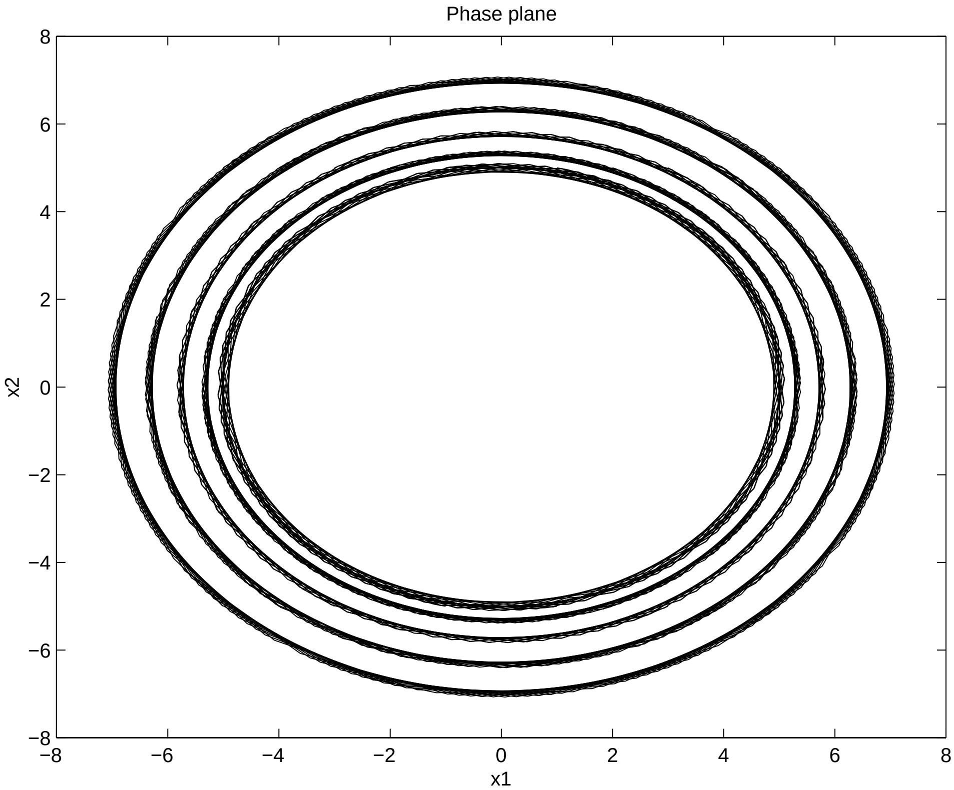


Figure 8: *Phase plane.* $\varepsilon = 1 \times 10^{-7}$

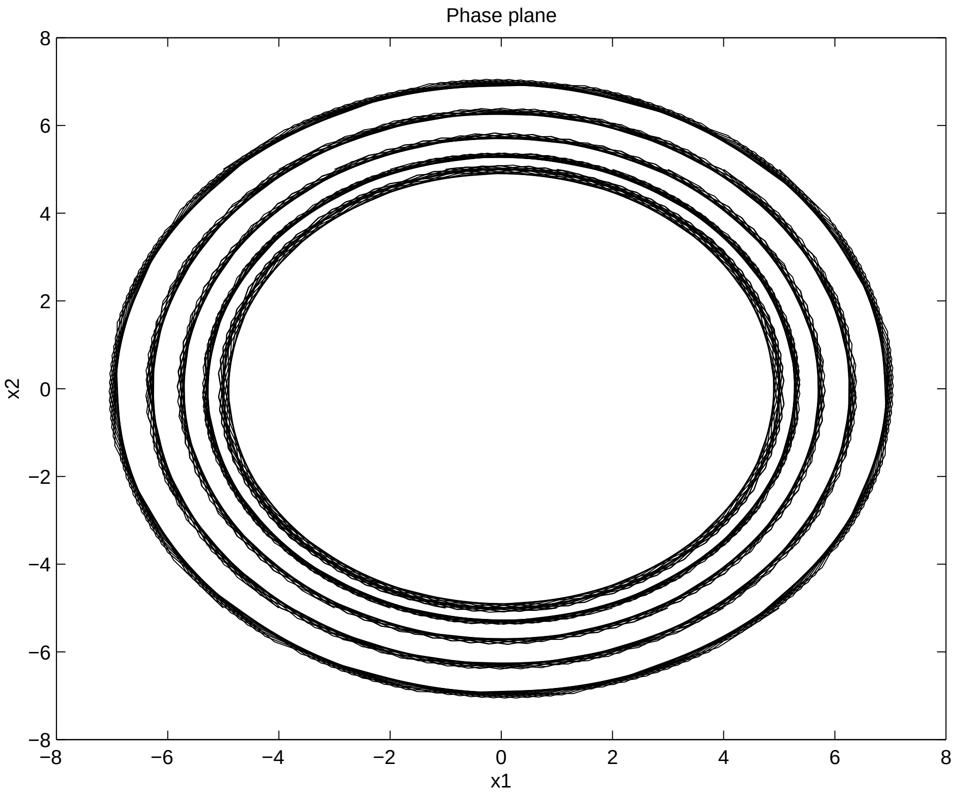
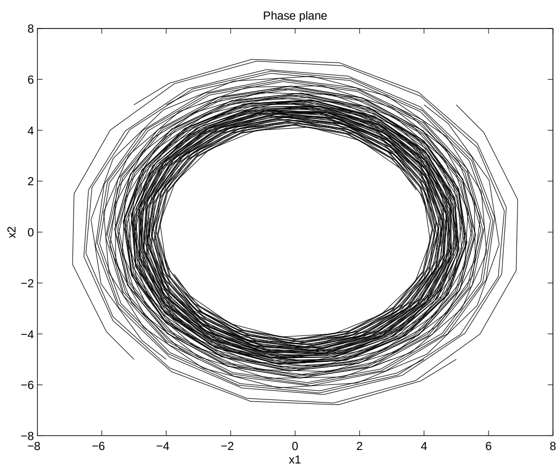
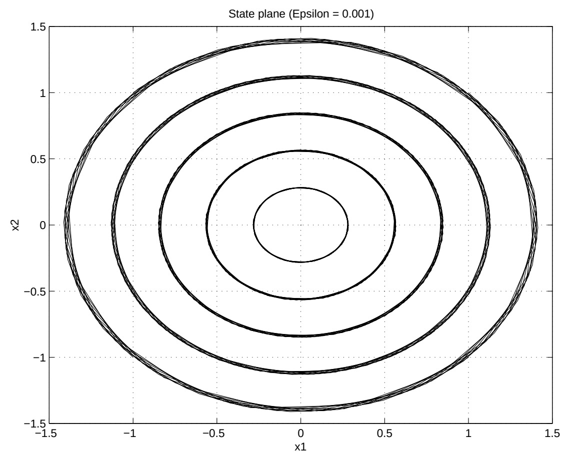


Figure 9: *Phase plane.* $\varepsilon = 1 \times 10^{-5}$

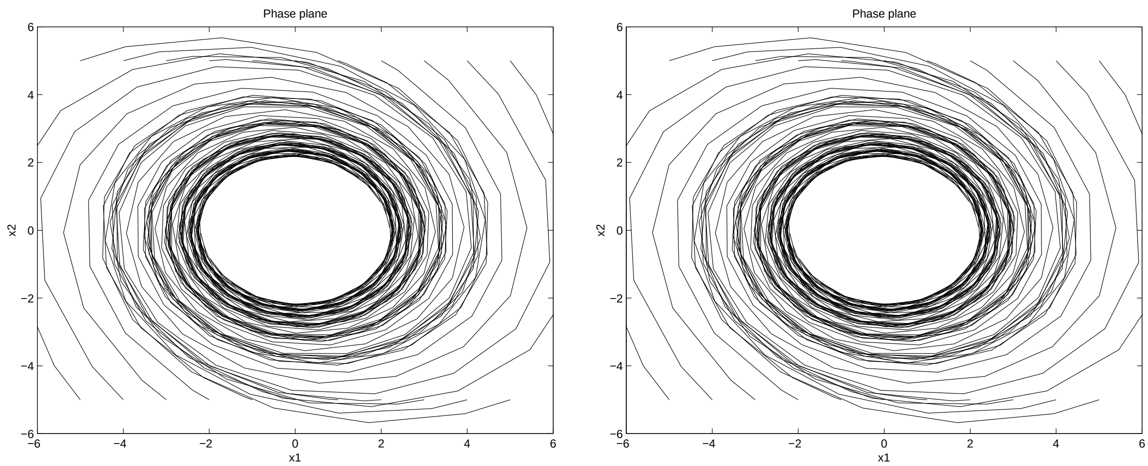


(a)



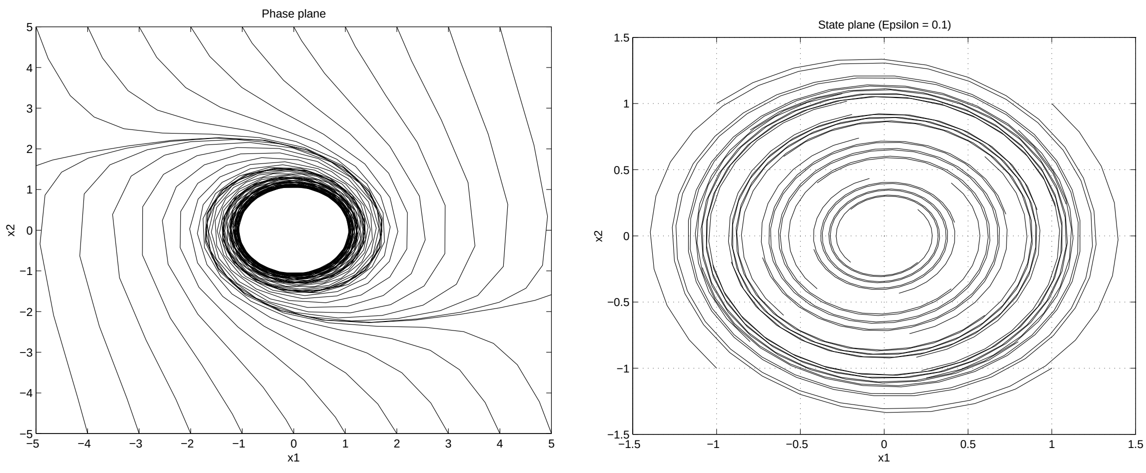
(b)

Figure 10: *Phase plane.* $\varepsilon = 0.001$ *No input.*



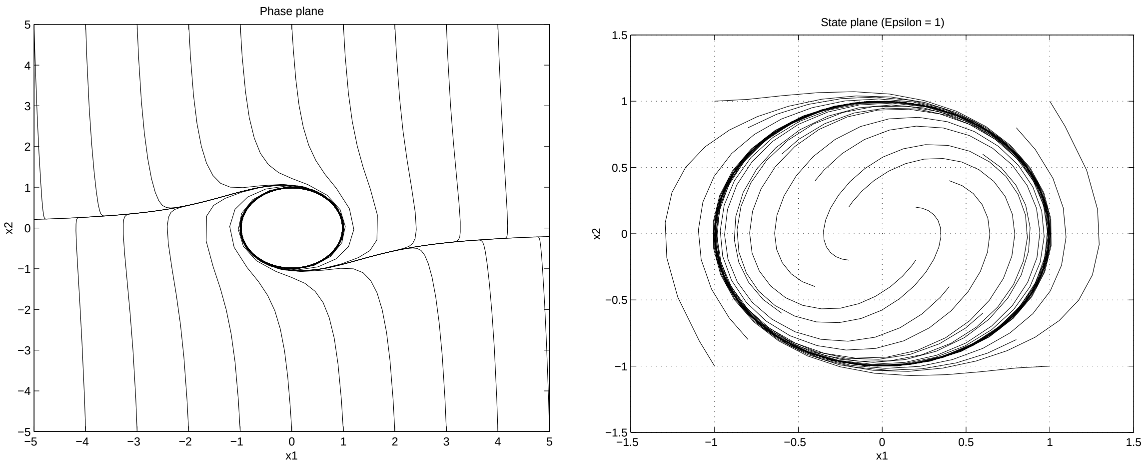
(a) (b)

Figure 11: *Phase plane.* $\varepsilon = 0.01$ *No input.*



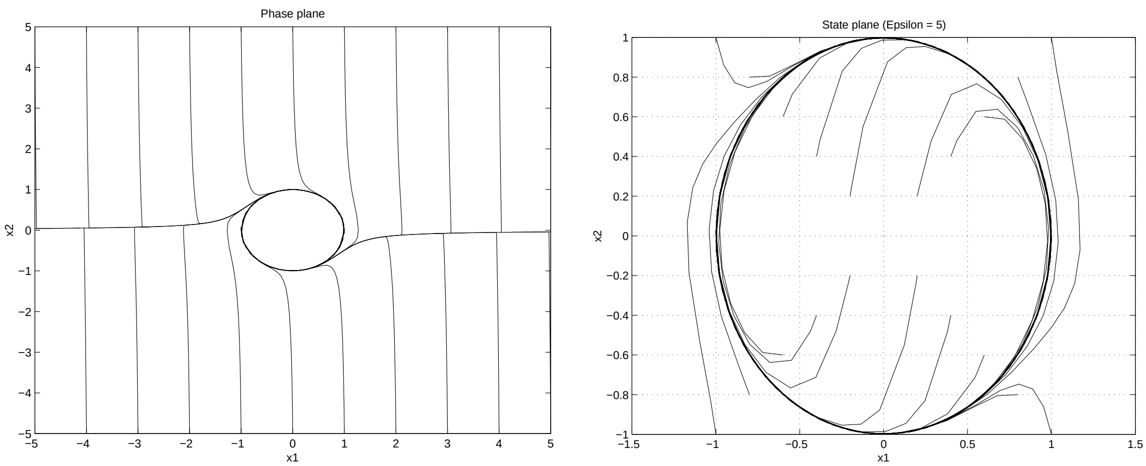
(a) (b)

Figure 12: *Phase plane.* $\varepsilon = 0.1$ *No input.*



(a) (b)

Figure 13: *Phase plane.* $\varepsilon = 1$ *No input.*



(a) (b)

Figure 14: *Phase plane.* $\varepsilon = 5$ *No input.*

Feedback of the signum function

Introduce an input

$$u = \operatorname{sgn}(x_1 + x_2)$$

and draw phase planes as shown in Figure 15 ($\varepsilon = 1 \times 10^{-7}$), Figure 16 ($\varepsilon = 0.001$), Figure 17 ($\varepsilon = 1$), Figure 18 ($\varepsilon = 5$).

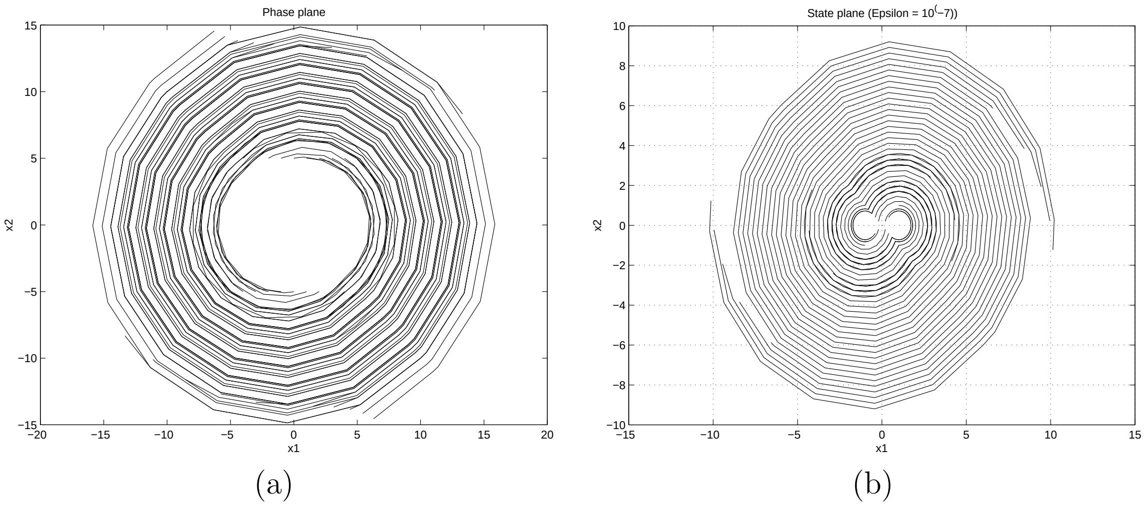
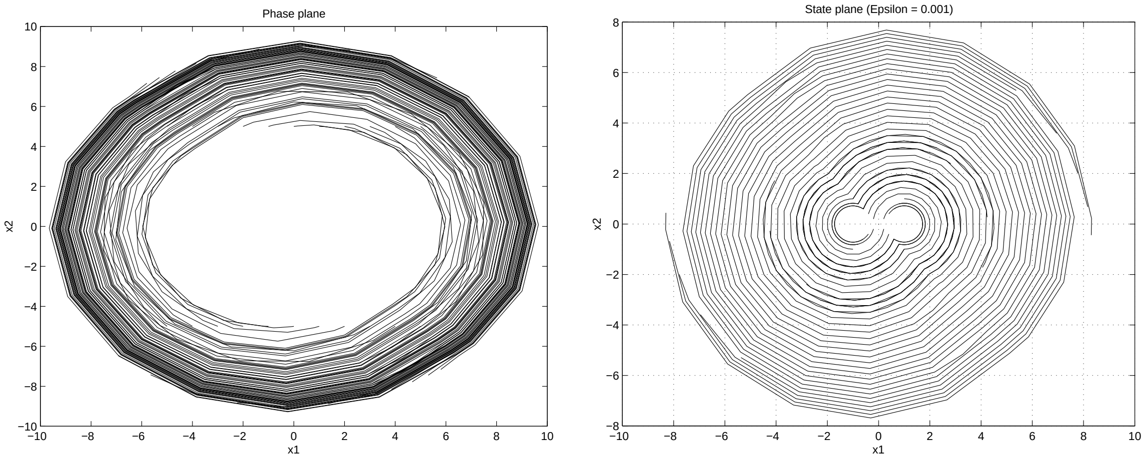


Figure 15: *Phase plane.* $\varepsilon = 1 \times 10^{-7}$ *with VS input.*



(a) (b)

Figure 16: *Phase plane.* $\varepsilon = 0.001$ *with VS input.*

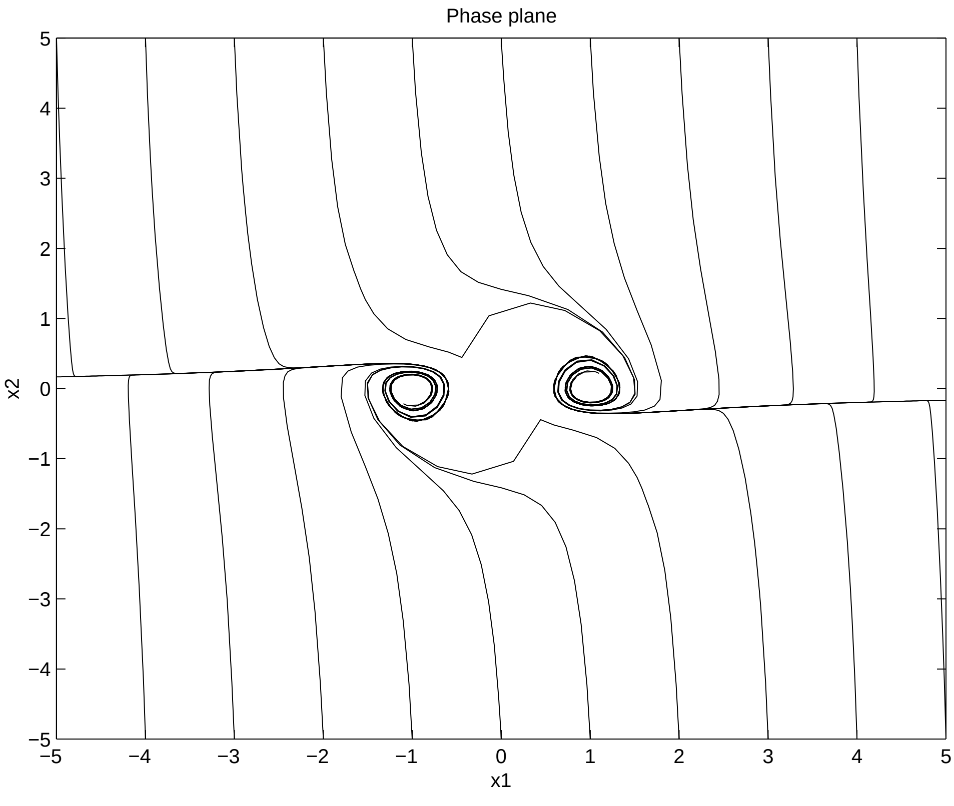


Figure 17: *Phase plane.* $\varepsilon = 1$ *with VS input.*

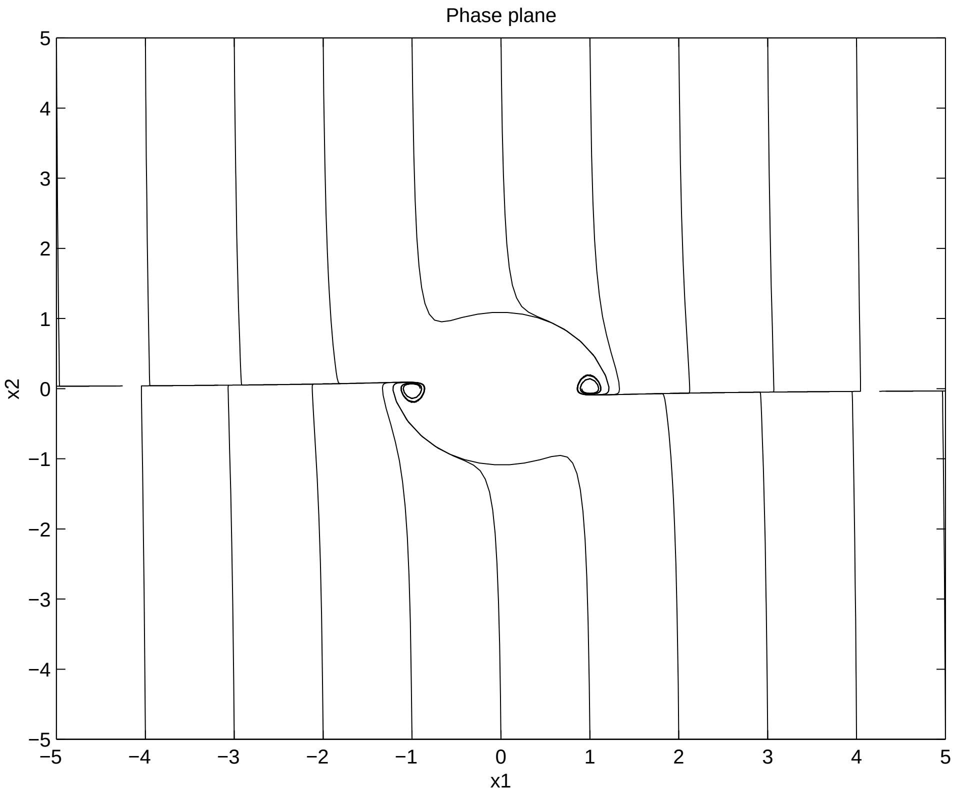


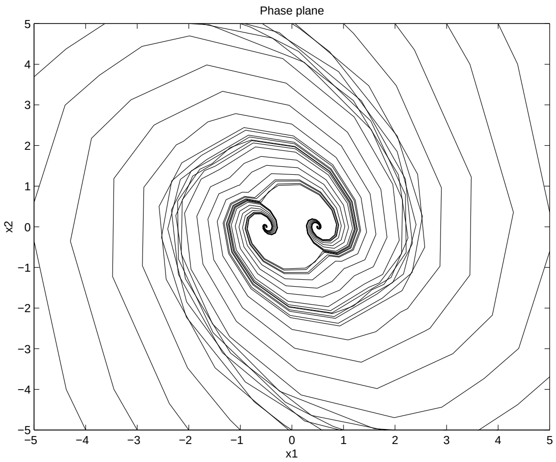
Figure 18: *Phase plane.* $\varepsilon = 5$ *with VS input.*

With feedback of signum function and states variables

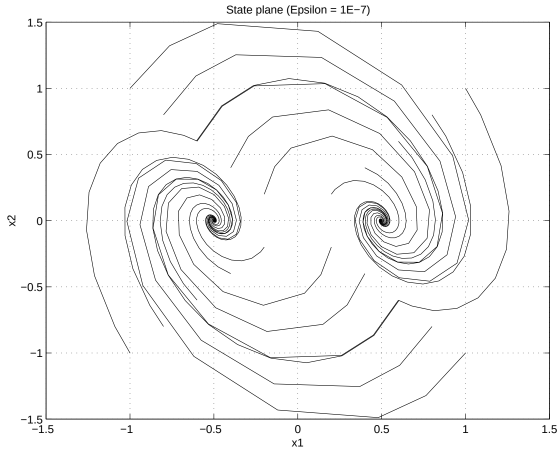
When subjected to an input

$$u = -(x_1 + x_2) + \operatorname{sgn}(x_1 + x_2)$$

this system produces phase planes as shown in Figure 19 ($\varepsilon = 1 \times 10^{-7}$), Figure 20 ($\varepsilon = 0.001$), Figure 21 ($\varepsilon = 1$), Figure 23 ($\varepsilon = 5$).



(a)



(b)

Figure 19: *Phase plane.* $\varepsilon = 1 \times 10^{-7}$ *with a VS input and feedback of state variables.*

An unstable case of positive state feedbacks plus a signum input

Let the input be

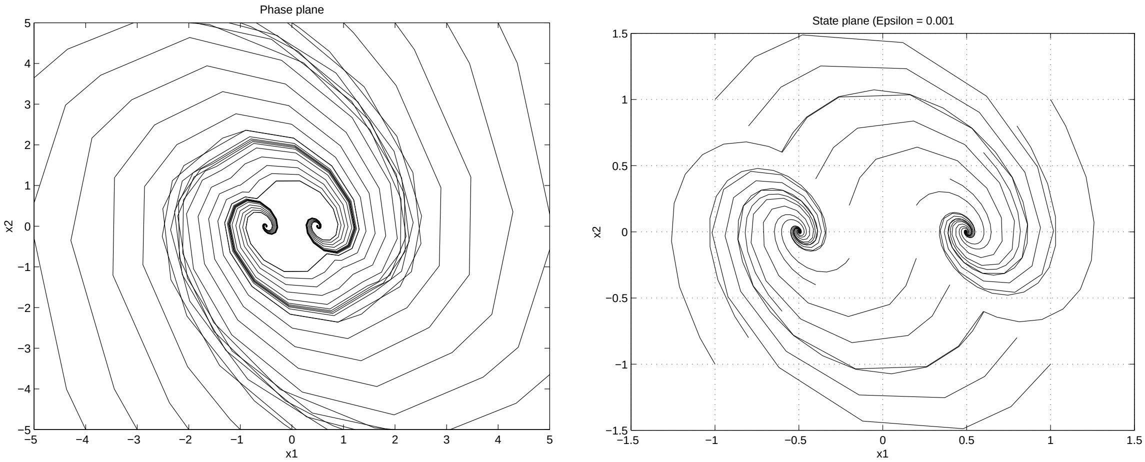
$$u = x_1 + x_2 + \operatorname{sgn}(x_1 + x_2) \tag{8}$$

which produce an unstable feedback system. Consider this case only briefly by showing the result in Figure 23 for various values of ε.

Discussion

From Figure 8 to Figure 14 the trajectories spirals toward a limit cycle of radius one. But when ε becomes very small ($\varepsilon \ll 1$) the rate which the trajectory moves toward this limit cycle becomes very small also, so that the result shown in Figure 8 and Figure 9 look like there are more than one limit cycle. Figure 12 to Figure 14 also shows that there are two hyper planes on either side of the limit cycle which the trajectory goes to and then slide toward the final destination. These hyperplanes become very distinct at $\varepsilon = 5$ as shown in Figure 14.

When $u = \operatorname{sgn}(x_1 + x_2)$ the response depends on the value of ε. With ε small $x_1 = \pm 1$ act as the two unstable foci as shown in Figure 15 ($\varepsilon = 10^{-7}$) and in Figure 16 ($\varepsilon = 0.001$), while with larger ε in the case of Figure 17 ($\varepsilon = 1$) or Figure 18 ($\varepsilon = 5$) the state plane is divided into two



(a) (b)

Figure 20: *Phase plane.* $\varepsilon = 0.001$ *with a VS input and feedback of state variables.*

regions. Each region has got its own focus and the trajectory in each region converges to a small limit cycle which circle around the focus.

When $u = -x_1 - x_2 + \text{sgn}\,(x_1 + x_2)$ Figure 19 to 21 show that there are two stable nodes at ± 0.5 for ε from very small upto $\varepsilon = 1$. When $\varepsilon = 5$ (Figure 23) there is a limit cycle circling around both stable points which used to be nodes. Figure 21 and 23 shows hyperplanes which trajectories approach first and then slide along it toward the nodes (in the case of Figure 21) or toward the limit cycle (in the case of Figure 23.)

Figure 23, though a stable feedback system, shows the effect of ε on the shape of the hyperplane and the trajectory which approaches it. Figure 23 from (a) to (f) are the result when ε becomes smaller and smaller. The trajectory in Figure 23 (f) looks like a sine wave.

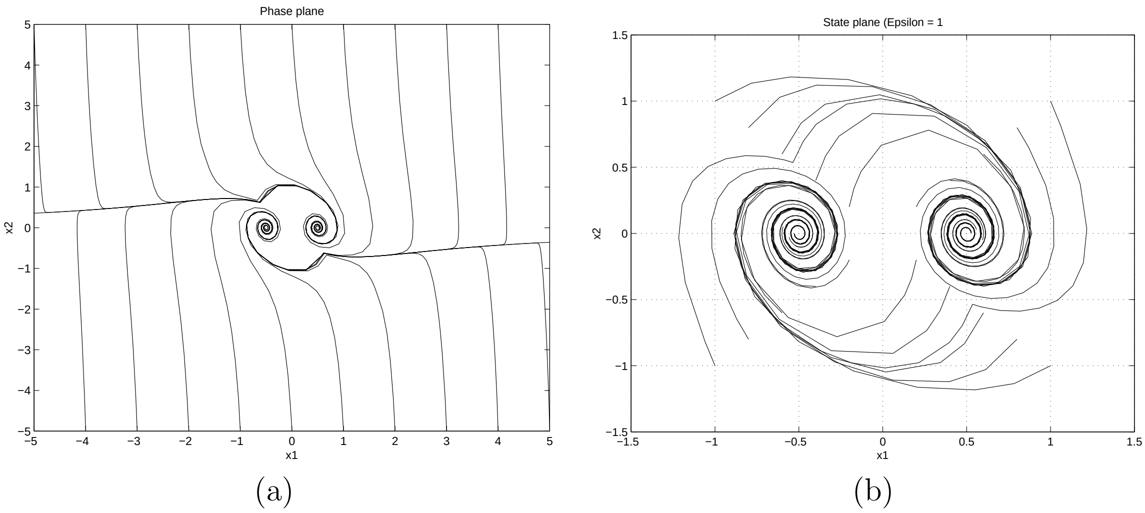


Figure 21: *Phase plane.* $\varepsilon = 1$ *with a VS input and feedback of state variables.*

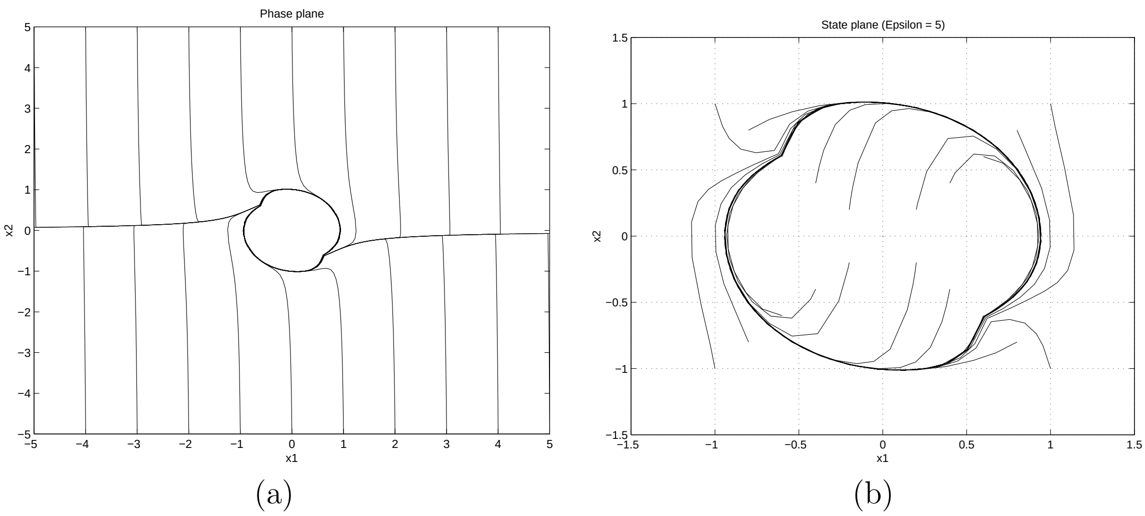


Figure 22: *Phase plane.* $\varepsilon = 5$ *with a VS input and feedback of state variables.*

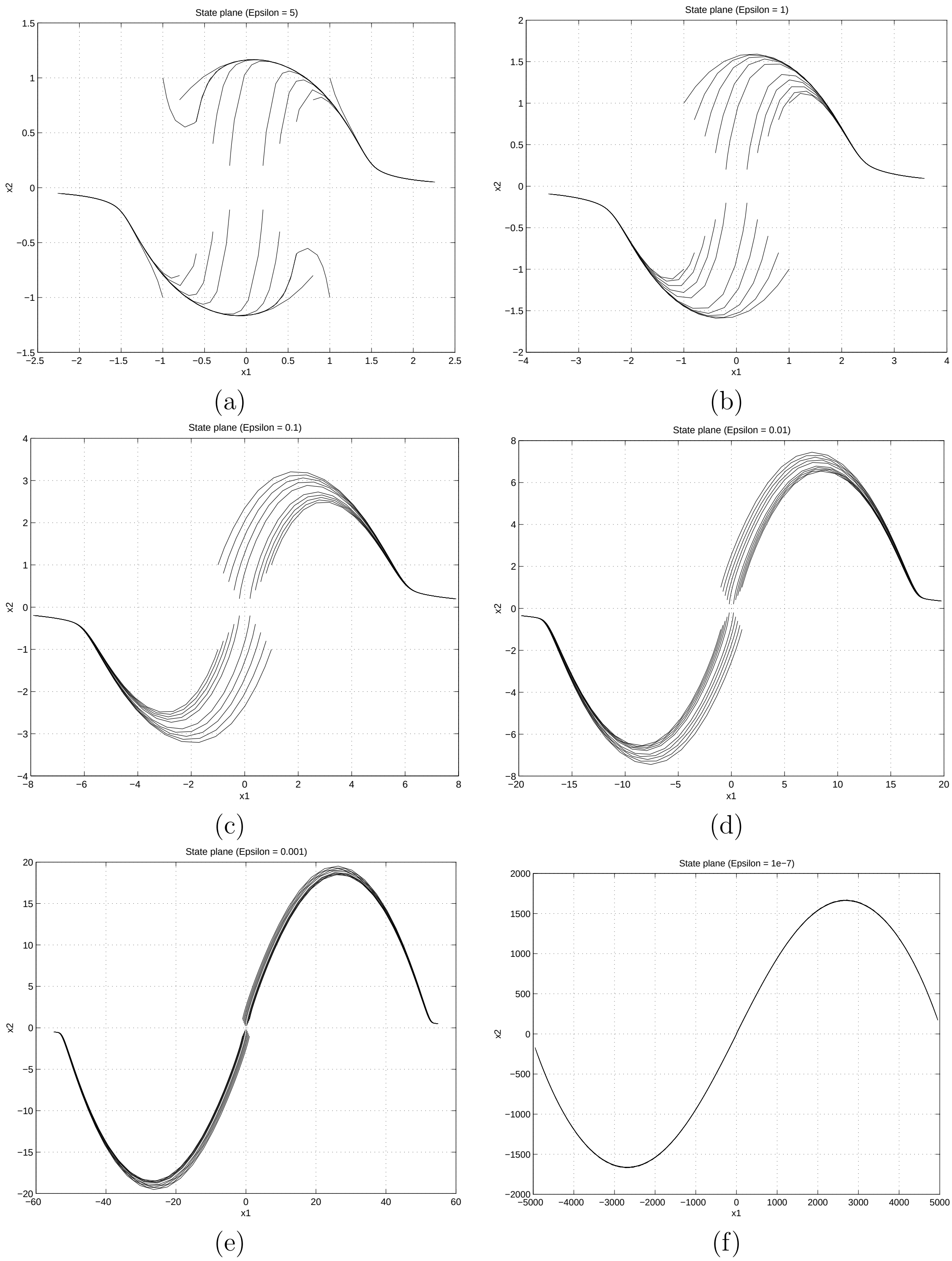


Figure 23: *Phase plane. (a)* $\varepsilon = 5$*, (b)* $\varepsilon = 1$*, (c)* $\varepsilon = 0.1$*, (d)* $\varepsilon = 0.01$*, (e)* $\varepsilon = 0.001$*, (f)* $\varepsilon = 10^{-7}$*, with a VS input and positive feedback of state variables.*

A system with a periodic orbit

21^{th} **April 1998**

Figure 24 shows the phase plane of the system [Kha96] (Exercise 7.1(1))

$$\dot{x_1} = x_2 \tag{9}$$
$$\dot{x_2} = -x_1 + x_2\left(1 - 3x_1^2 - 2x_2^2\right) + u, \tag{10}$$

with $u = 0$.

Some knowledge about this system

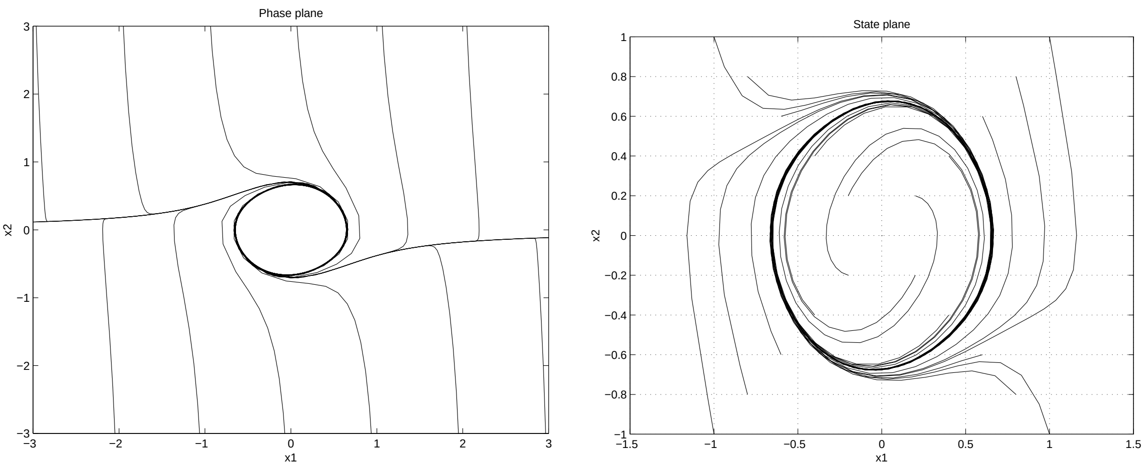
The equilibrium point is the origin. (ie the solution of $f(x) = 0$ when $\dot{x} = f(x)$) The Jacobian matrix of this system is

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\left[\begin{array}{cc} 0 & 1 \\ -1 - 6x_1x_2 & 1 - 3x_1^2 - 6x_2^2 \end{array}\right]\right|_{x=0} = \left[\begin{array}{cc} 0 & 1 \\ -1 & 1 \end{array}\right]. \tag{11}$$

Then the Eigenvalues are $0.5 \pm \jmath 0.866$ which means that the equilibrium point is an unstable focus.

Phase plot of the system

The phase plane of this system has got a periodic orbit.



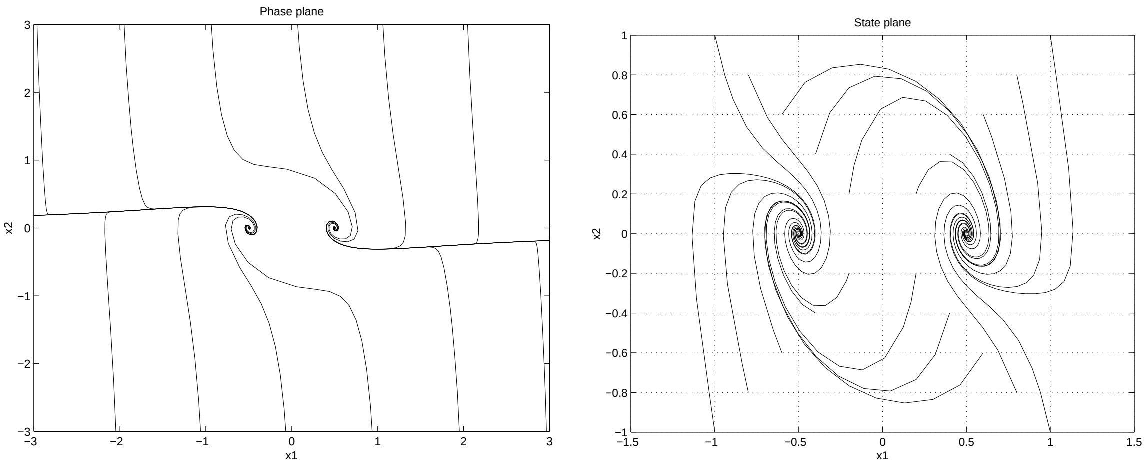
(a) (b)

Figure 24: *Phase plane. No input.*

With feedback of a signum function and state variables

Figure 25 shows a phase plane of this system with a feedback of s.v. (state variables) and a VS control, ie.

$$u = -x_1 - x_2 + \mathrm{sgn}(x_1 + x_2).$$



(a) (b)

Figure 25: *Phase plane of the system with a feedback of state variables and a VS input.*

With feedback of a signum function

Figure 26 shows a phase plane of this system with a VS input only, ie.

$$u = \mathrm{sgn}(x_1 + x_2).$$

Discussion

Figure 24 shows that the origin is an equilibrium point which acts as an unstable focus, and that there exists a limit cycle as well as two hyperplanes leading trajectories into this limit cycle. Both Figure 25 and Figure 26 show that there are two equilibrium points acting as nodes. It is interesting to note that in both of these two figures trajectories starting from the hyperplane will goto either one of these nodes.

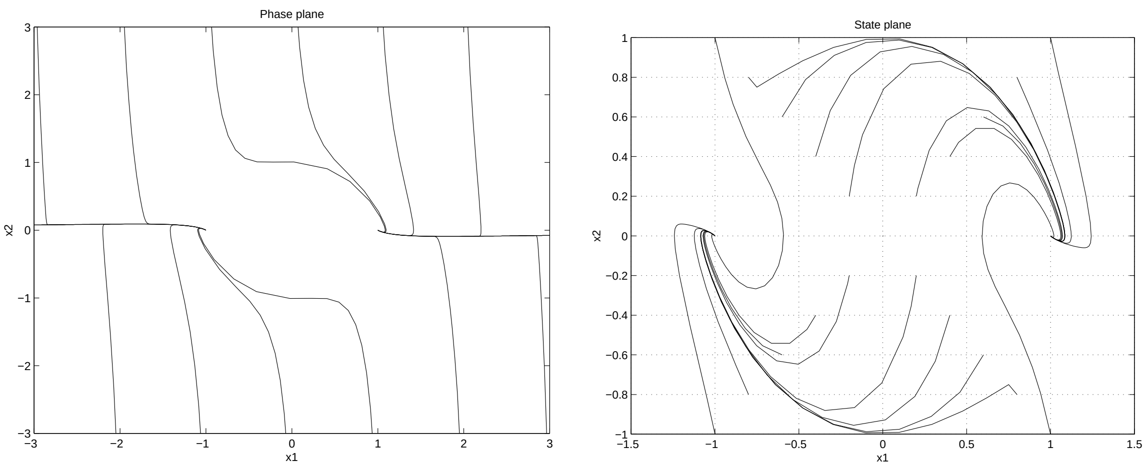


Figure 26: *Phase plane of the system with a VS input only.*

Another system with a periodic orbit

Figure 27 shows a phase plane of the system [Kha96] (Exercise 7.1(2))

$$\dot{x_1} = x_2 \tag{12}$$
$$\dot{x_2} = -x_1 + x_2 - 2(x_1 + 2x_2)x_2^2 + u, \tag{13}$$

with $u = 0$.

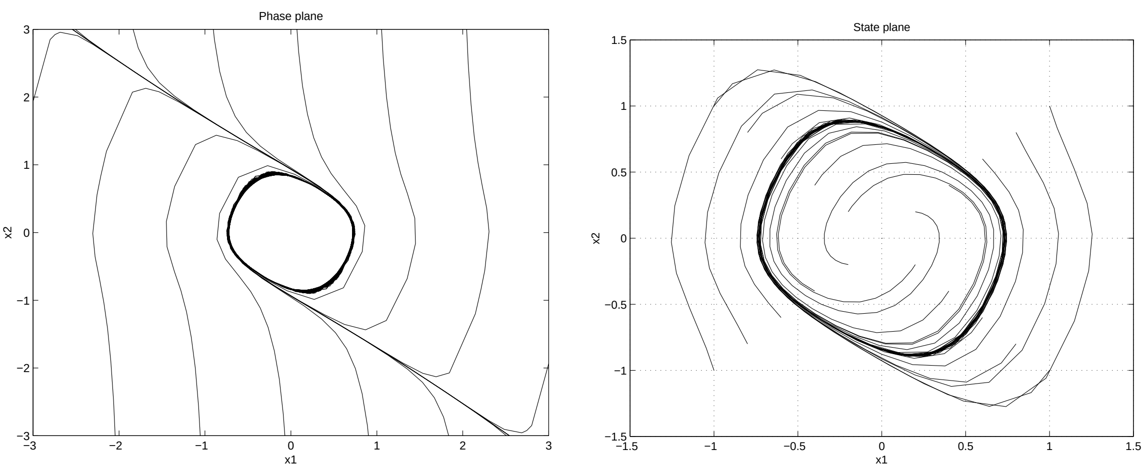
Some knowledge about the system

The solution of $f(x) = 0$ when $\dot{x} = f(x)$ gives the equilibrium point to be at the origin. The Jacobian matrix of this system can be found by

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\left[\begin{array}{cc} 0 & 1 \\ -1 - 2x_2^2 & 1 - 4x_2^2 - 4x_2\,(x_1 + 2x_2) \end{array}\right]\right|_{x=0} = \left[\begin{array}{cc} 0 & 1 \\ -1 & 1 \end{array}\right]. \tag{14}$$

Then the Eigenvalues are $0.5 \pm \jmath 0.866$ which means that the equilibrium point is an unstable focus.

Phase plot of the system



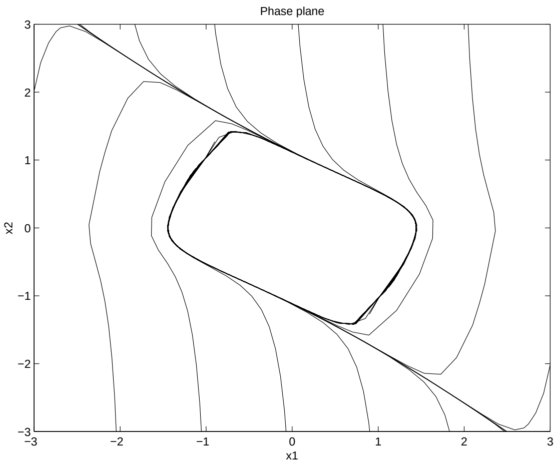
(a) (b)

Figure 27: *Phase plane. No input.*

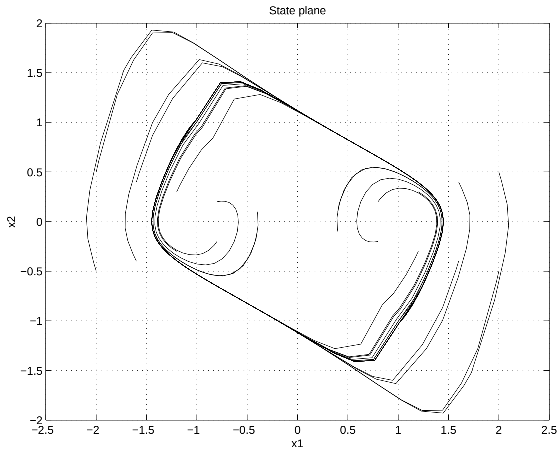
Sample response with a variable structure input

Figure 28 shows the phase plane when the system is subjected to

$$u = \mathrm{sgn}(x_1 + x_2).$$



(a)



(b)

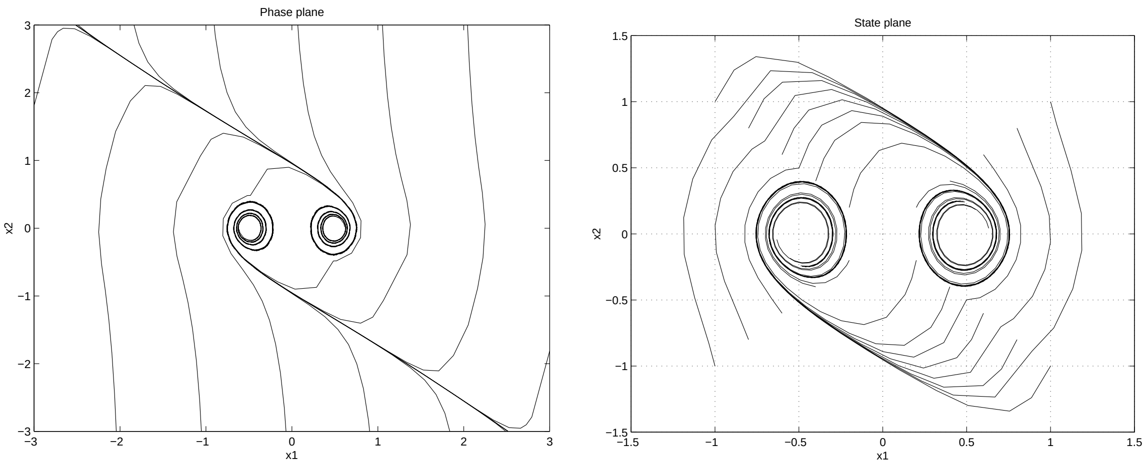
Figure 28: *Phase plane with a VS input.*

Sample response with feed back of states and a variable structure input

Figure 29 shows the phase plane when the system is subjected to

$$u = -(x_1 + x_2) + \mathrm{sgn}(x_1 + x_2)$$

respectively.



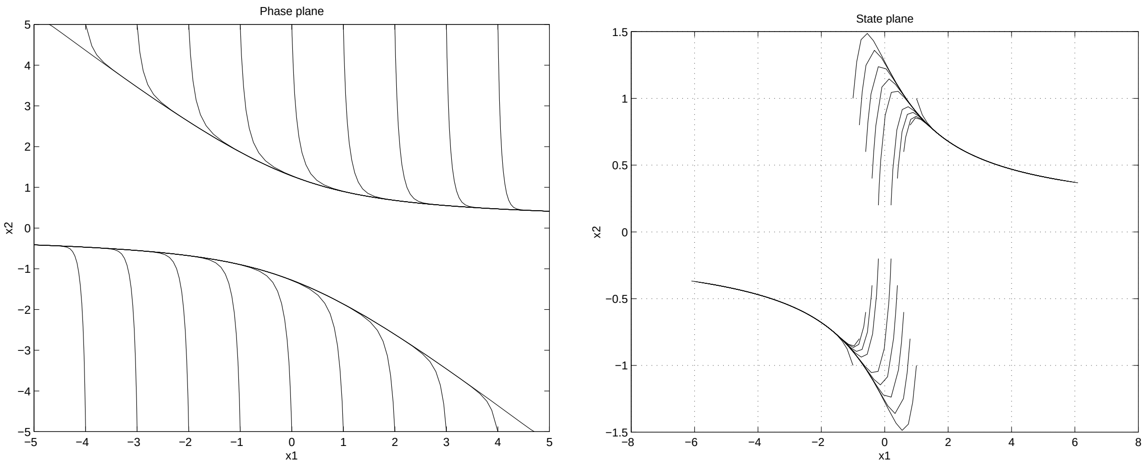
(a) (b)

Figure 29: *Phase plane with a negative feedback of state variables and a VS input.*

Figure 30 shows the phase plane of this system subjected to

$$u = x_1 + x_2 + \text{sgn}(x_1 + x_2).$$

When $u = x_1 + x_2$ the phase plane is similar to Figure 30.



(a) (b)

Figure 30: *Phase plane with a positive feedback of state variables and a VS input.*

Discussion

Both Figure 27, 28 and 29 shows that there is a limit cycle as well as hyperplanes which lead to the limit cycle. Without an input Figure 27 shows that the equilibrium point at the origin is an unstable focus. Figure 28 shows that with a signum function input there are now three equilibrium points instead of one, the origin has changed into being an unstable node while the other two which are located at $x_1 = \pm 1, x_2 = 0$ are now the unstable foci.

Figure 28 also shows that the hyperplanes of the original system are still there while on the new hyperplane introduced via variable structure method ($s = x_1 + x_2 = 0$) the trajectories are deflected.

Figure 29 shows things similar to those which were described above for Figure 28. However for Figure 29 the deflection of trajectories at the hyperplane is more prominent and the two unstable foci are now at $(\pm 0.5, 0)$ instead of at $(\pm 1, 0)$. (also the foci becomes more prominent in a way that trajectories wind around them more prominently)

The unstable feedback shown in Figure 30 also has two hyperplanes.

DC motor

20^{th} **April 1998**

An armature-controlled DC motor [Kha96] is shown here as Figure 31. The corresponding state

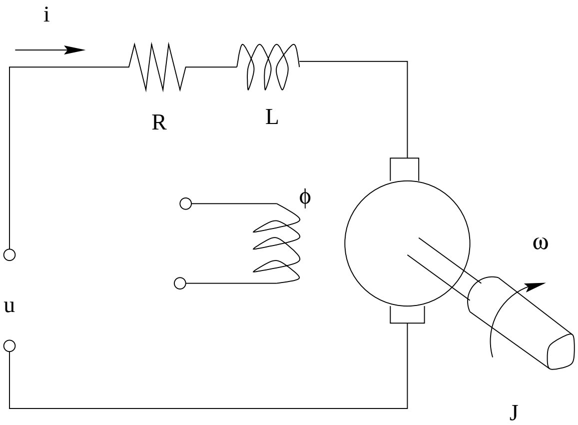


Figure 31: *Armature-controlled DC motor.*

equation is

$$J\frac{d\omega}{dt} = ki \tag{15}$$

$$L\frac{di}{dt} = -k\omega - Ri + u \tag{16}$$

where i, u, R, L are the armature current, voltage, resistance and inductance, J is the moment of inertia, ω the angular speed. The constant excitation flux ϕ results in a torque ki and a back e.m.f. $k\omega$

Then the state equation of a DC motor with all the dimension made dimensionless is

$$T_m\frac{d\omega_r}{dt} = i_r \tag{17}$$

$$T_e\frac{di_r}{dt} = -\omega_r - i_r + u_r, \tag{18}$$

where

$$\omega_r = \frac{\omega}{\Omega},\ i_r = \frac{iR}{k\Omega},\ u_r = \frac{u}{k\Omega},\ T_m = \frac{JR}{k^2},\ T_e = \frac{L}{R}$$

T_m is teh mechanical time constant, T_e the electrical time constant and $T_m \gg T_e$. Rewrite the state equation as

$$\frac{d\omega_r}{dt_r} = i_r \tag{19}$$

$$\varepsilon\frac{di_r}{dt_r} = -\omega_r - i_r + u_r, \tag{20}$$

where

$$t_r = \frac{t}{T_m}, \quad \varepsilon = \frac{T_e}{T_m} = \frac{Lk^2}{JR^2}$$

Some knowledge about the DC motor system equation

At the equilibrium point

$$\frac{d\omega_r}{dt_r} = 0, \quad \text{and} \quad \frac{di_r}{dt_r} = 0$$

which gives the origin $(i_r = 0, \omega_r = 0)$ as the only equilibrium point. Moving ε to the right-hand side in the second equation we have

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \end{bmatrix}$$

which gives the eigenvalues as

$$\frac{1}{2\varepsilon}\left(-1 \pm \sqrt{(1-4\varepsilon)}\right).$$

Seeing that the eigenvalues depend upon the value of ε, plot Figure 32 to show the values visually as ε varies. The intrepretation of Figure 32 is Table 2.

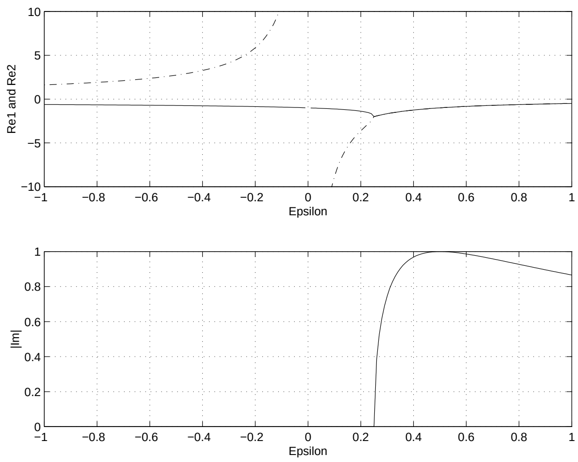


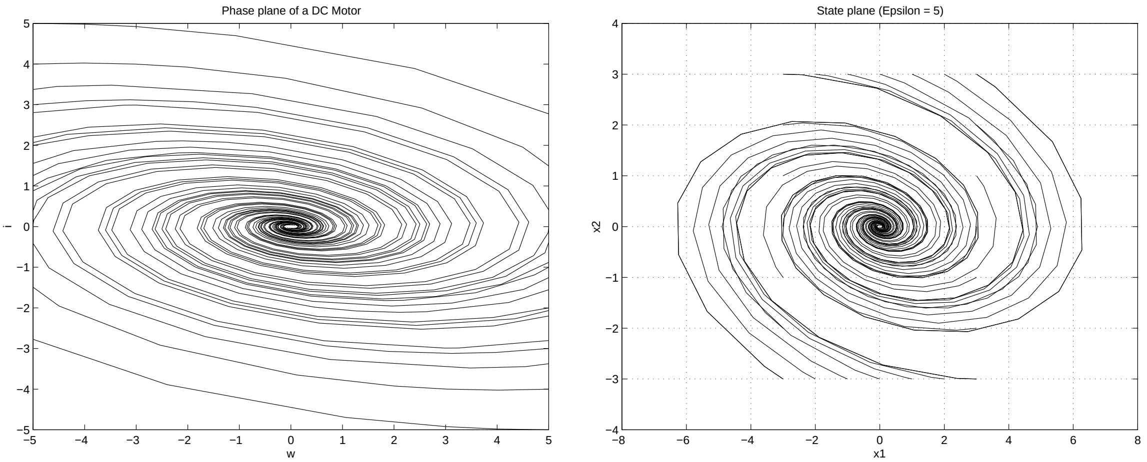
Figure 32: *The eigenvalues as a function of* ε.

Response with various values of ε

Figure 33 shows the phase plane of this system when $\varepsilon = 5$, Figure 34 $\varepsilon = 2$, Figure 35 $\varepsilon = 1$, Figure 36 $\varepsilon = 0.1$, Figure 37 $\varepsilon = 0.01$.

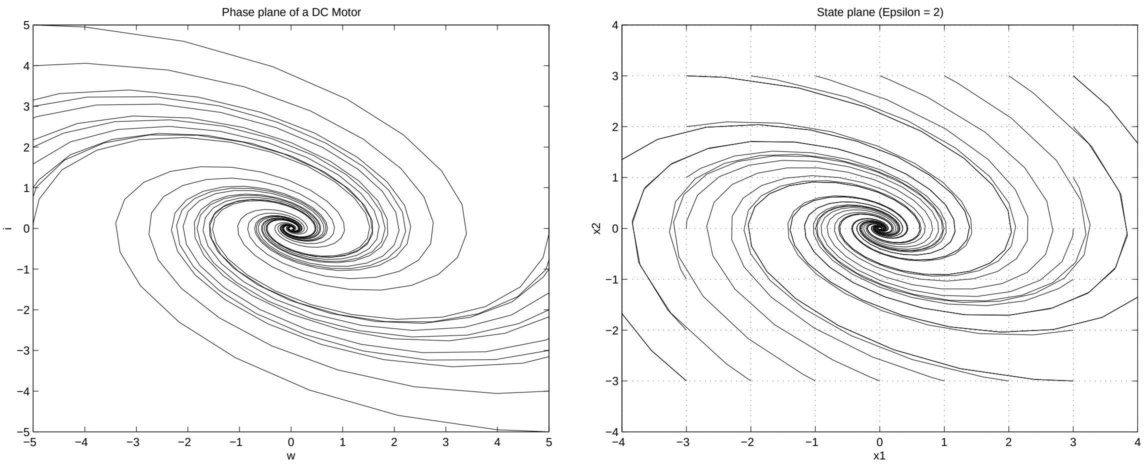
Values of ε	Characteristic of the trajectories
$(-\infty, 0)$	saddle point
0	system degenerated
$(0, 0.25]$	stable node
$(0.25, \infty)$	stable focus

Table 2: *Characteristic of the trajectories for DC motor system equation as function of* ε



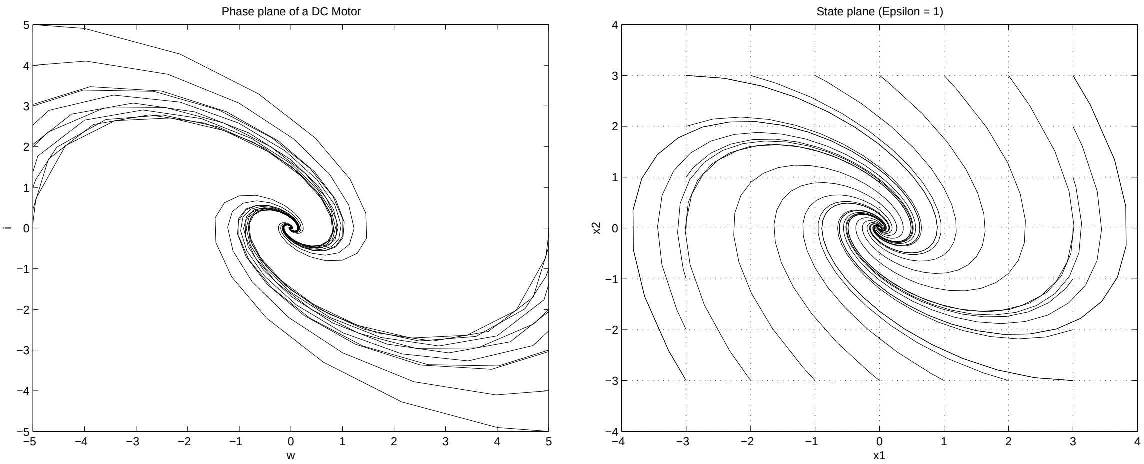
(a) (b)

Figure 33: *Phase plane of a DC motor,* $\varepsilon = 5$



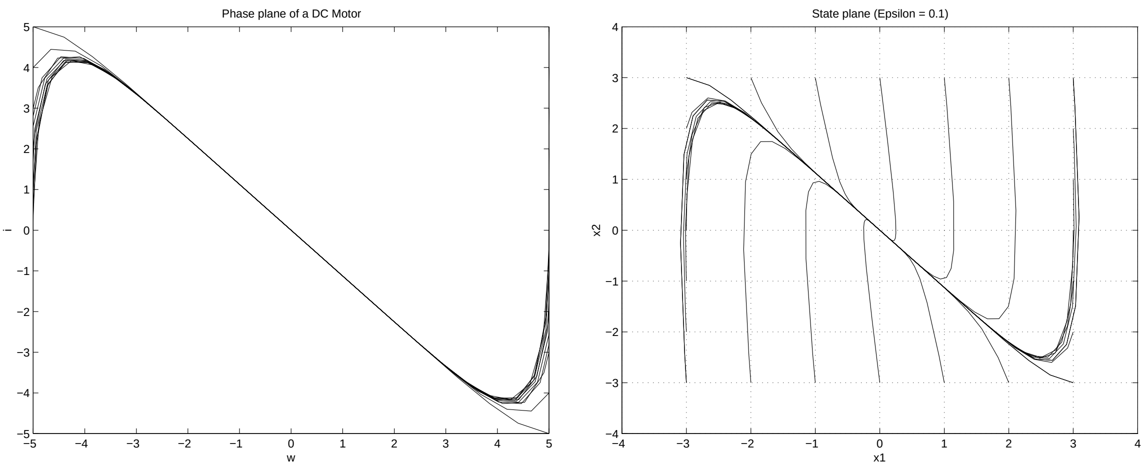
(a) (b)

Figure 34: *Phase plane of a DC motor,* $\varepsilon = 2$



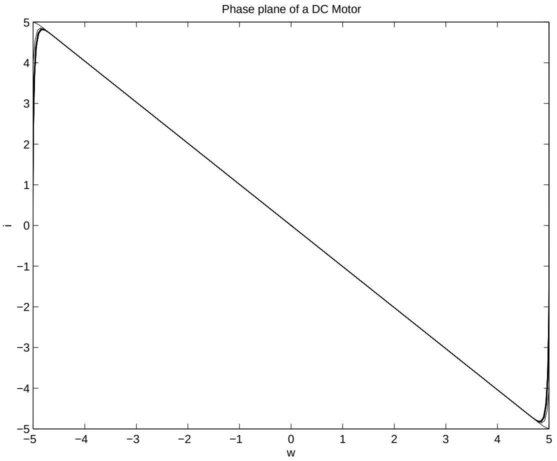
(a) (b)

Figure 35: *Phase plane of a DC motor,* $\varepsilon = 1$

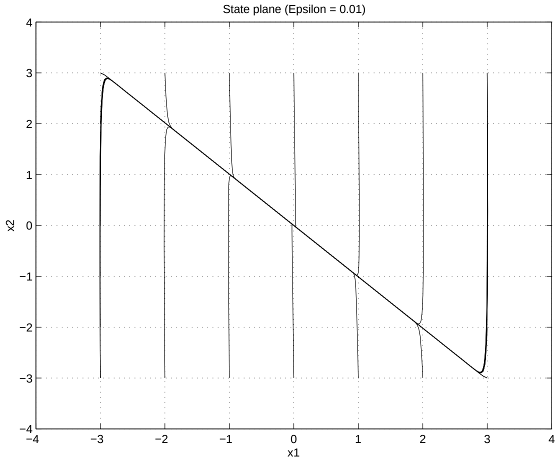


(a) (b)

Figure 36: *Phase plane of a DC motor,* $\varepsilon = 0.1$



(a)



(b)

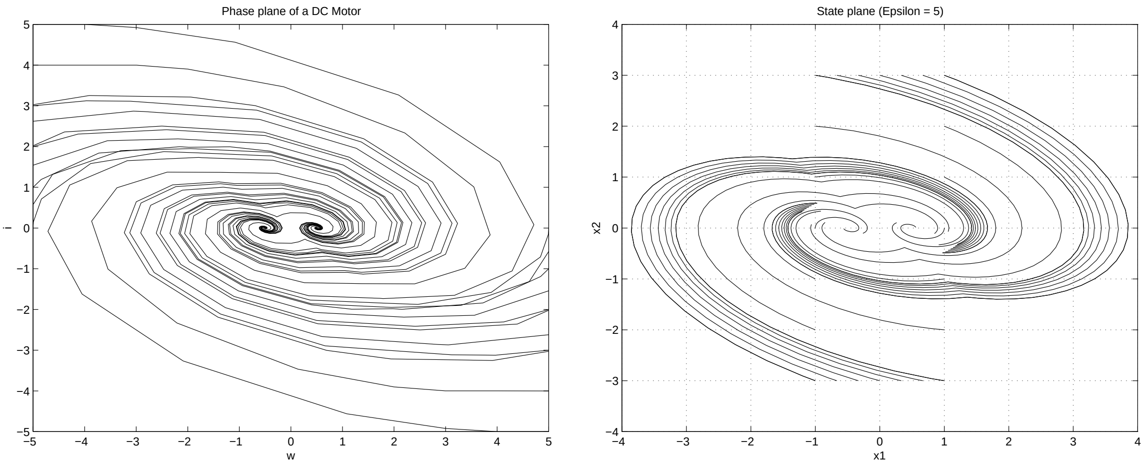
Figure 37: *Phase plane of a DC motor*, $\varepsilon = 0.01$

Sample plots for various ε of the DC motor with states feedback and VS in the input

Next with state feedbacks and a variable structure control, simulate the same system with an input

$$u_r = -i_r - w_r + \mathrm{sgn}(i_r + w_r).$$

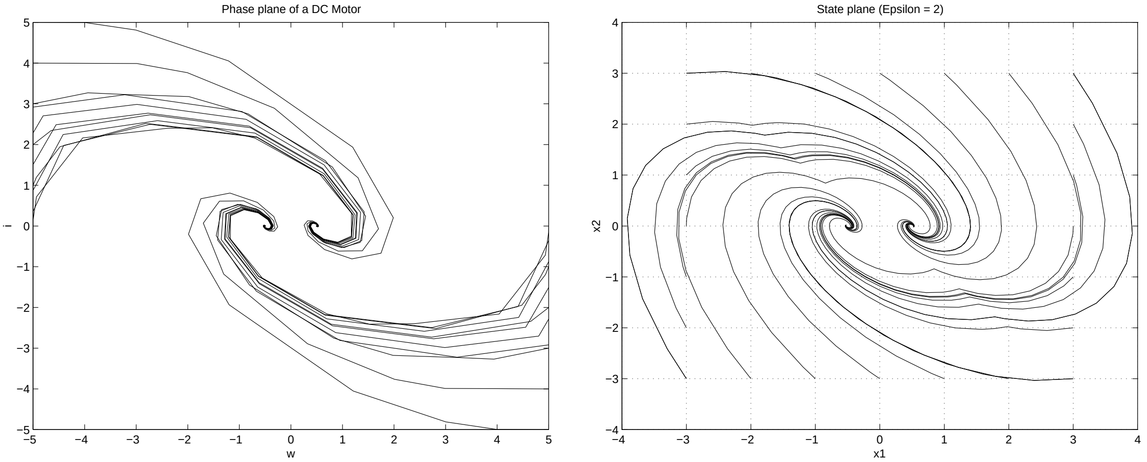
The results are shown in Figure 38, Figure 39, Figure 40, Figure 41 and Figure 42.



(a) (b)

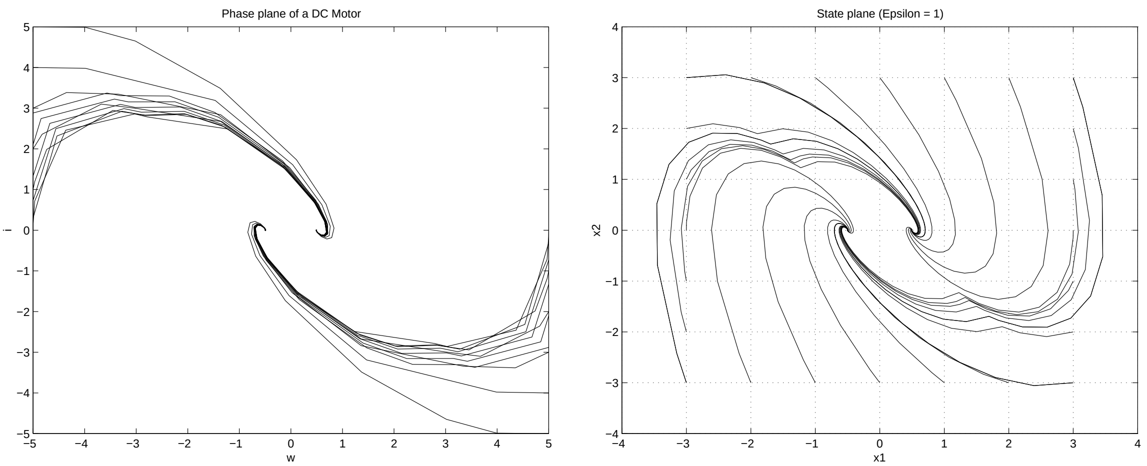
Figure 38: *Phase plane of a DC motor, $\varepsilon = 5$. With state feedbacks and a VS input.*

Discussion



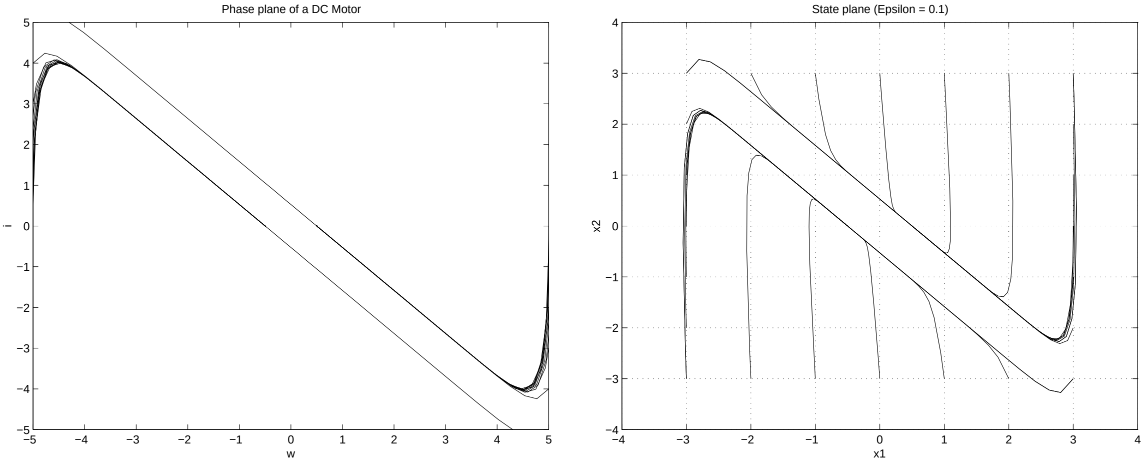
(a) (b)

Figure 39: *Phase plane of a DC motor,* $\varepsilon = 2$*. With state feedbacks and a VS input.*



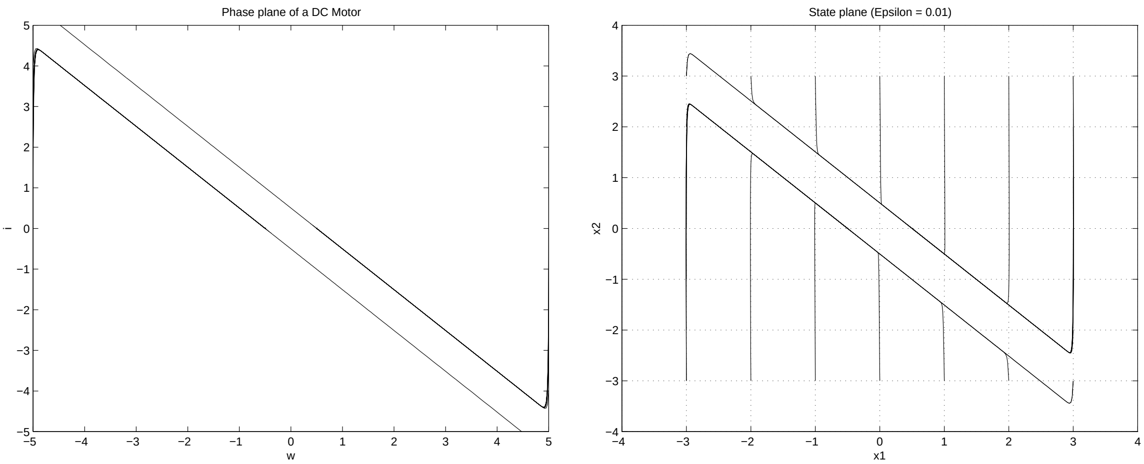
(a) (b)

Figure 40: *Phase plane of a DC motor,* $\varepsilon = 1$*. With state feedbacks and a VS input.*



(a) (b)

Figure 41: *Phase plane of a DC motor, $\varepsilon = 0.1$. With state feedbacks and a VS input.*



(a) (b)

Figure 42: *Phase plane of a DC motor, $\varepsilon = 0.01$. With state feedbacks and a VS input.*

Van der Pol

16^{th} **April 1998**
This time consider the Van der Pol system represented in a state equation form as

$$\begin{aligned} \dot{x} &= z \quad (21) \\ \varepsilon \dot{z} &= -x + z - \frac{1}{3}z^3. \quad (22) \end{aligned}$$

Some knowledge about the Van der Pol system

The only equilibrium point is at the origin. $\left(z = 0, \text{ and } -x + z - \frac{1}{3}z^3 = 0\right)$ The Jacobian matrix is

$$\begin{bmatrix} 0 & 1 \\ -\frac{1}{\varepsilon} & \frac{1}{\varepsilon} - \frac{3z^2}{\varepsilon} \end{bmatrix}$$

And the Jacobian matrix evaluated at the equilibrium point is

$$A = \left.\begin{bmatrix} 0 & 1 \\ -\frac{1}{\varepsilon} & \frac{1}{\varepsilon} - \frac{3z^2}{\varepsilon} \end{bmatrix}\right|_{x=0, z=0} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\varepsilon} & \frac{1}{\varepsilon} \end{bmatrix}.$$

The eigenvalues of the system at the equilibrium point is then

$$\frac{1}{2\varepsilon}\left[1 \pm \sqrt{1 - 4\varepsilon}\right]$$

which can be shown graphically as Figure 43, the description of which is written in Table 3.

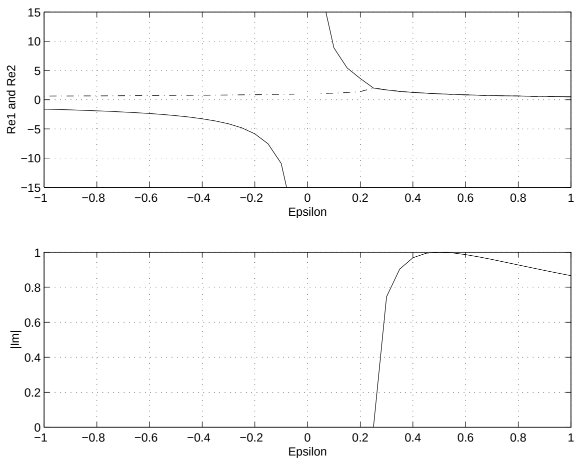


Figure 43: *Eigenvalues movements when ε varies for the Van der Pol system.*

Examples of response of Van der Pol with different values of ε

Plot the graph of the state variables both in time domain and as a parametric plot for $\varepsilon = 1$ (Figure 44 and 45), $\varepsilon = 0.1$ (Figure 46 and 47) and $\varepsilon = 1$ (Figure 48 and 49).

Values of ε	nature of the origin
$(-\infty, 0)$	saddle point
0	degenerated system
$(0, 0.25]$	unstable node
$(0.25, \infty)$	unstable focus

Table 3: *Characteristics of trajectories of the Van der Pol system as related to* ε*.*

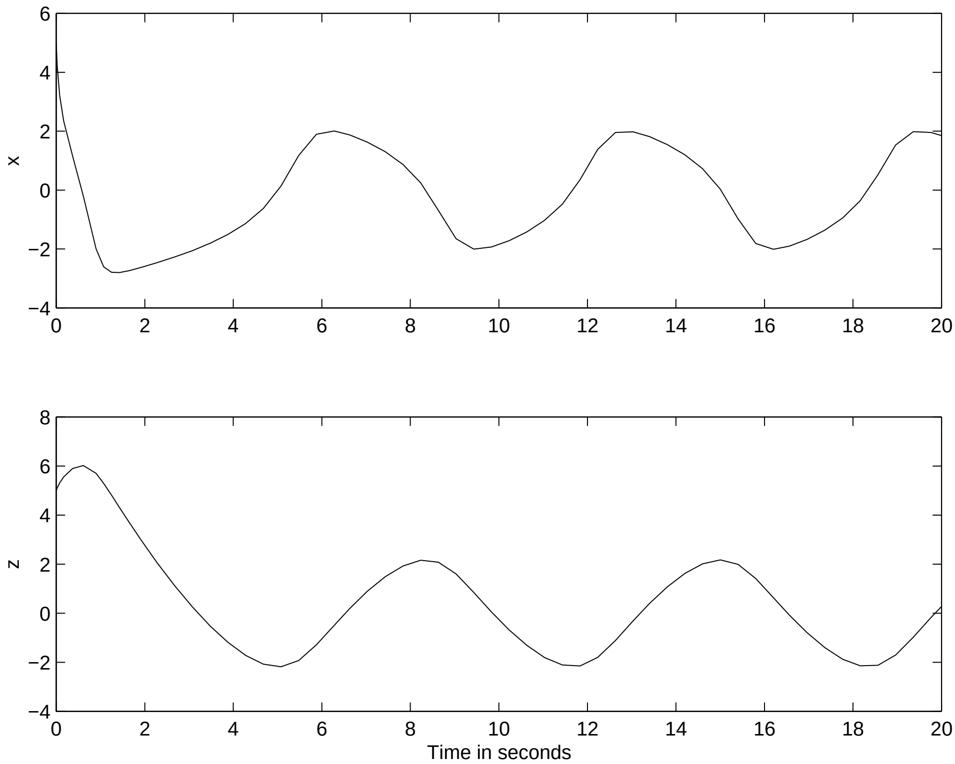
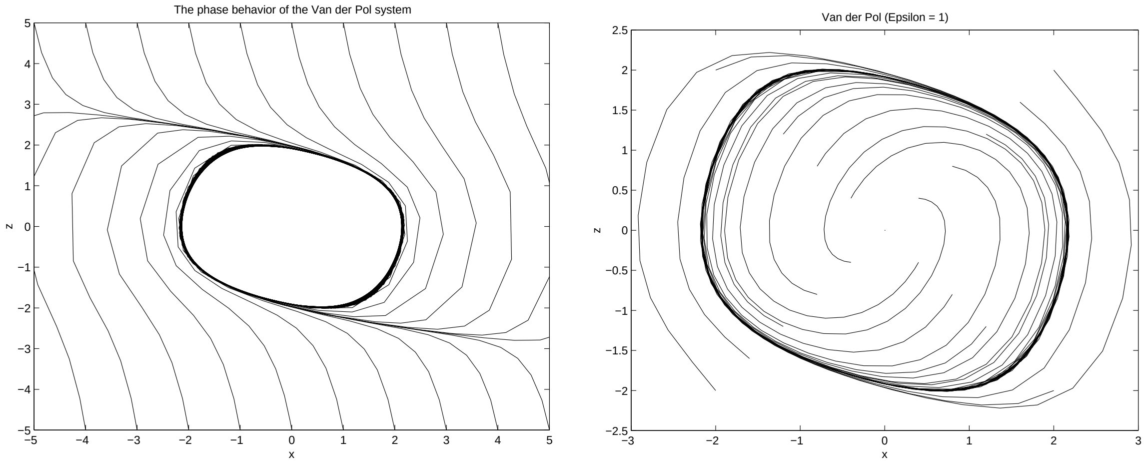


Figure 44: *State variables of a Van der Pol system,* $\varepsilon = 1$*.*



(a) (b)

Figure 45: *Phase portrait of a Van der Pol system when* $\varepsilon = 1$.

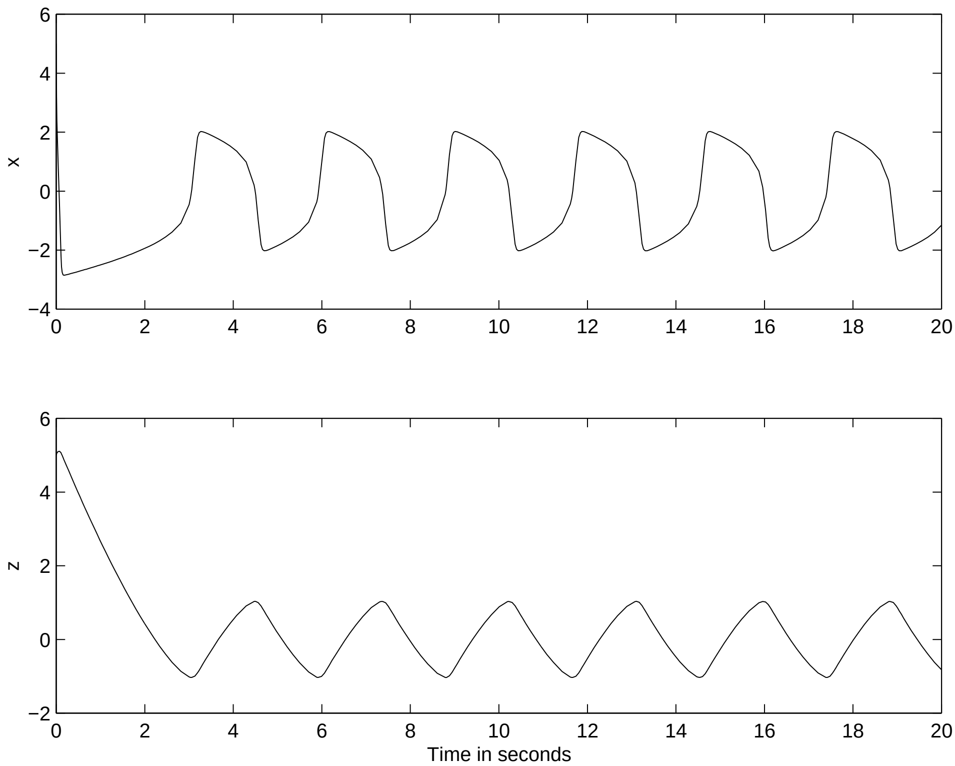
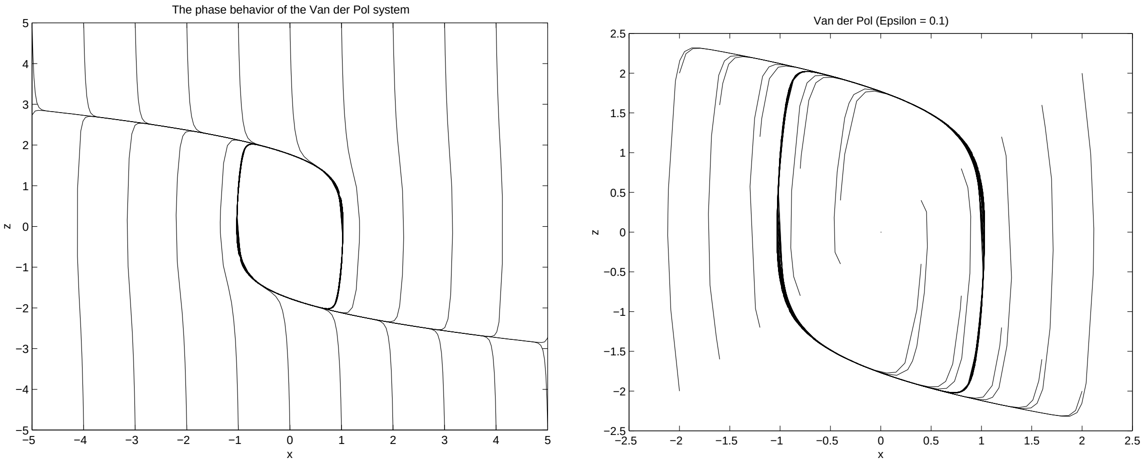


Figure 46: *State variables of a Van der Pol system,* $\varepsilon = 0.1$.



(a) (b)

Figure 47: *Phase portrait of a Van der Pol system when* $\varepsilon = 0.1$.

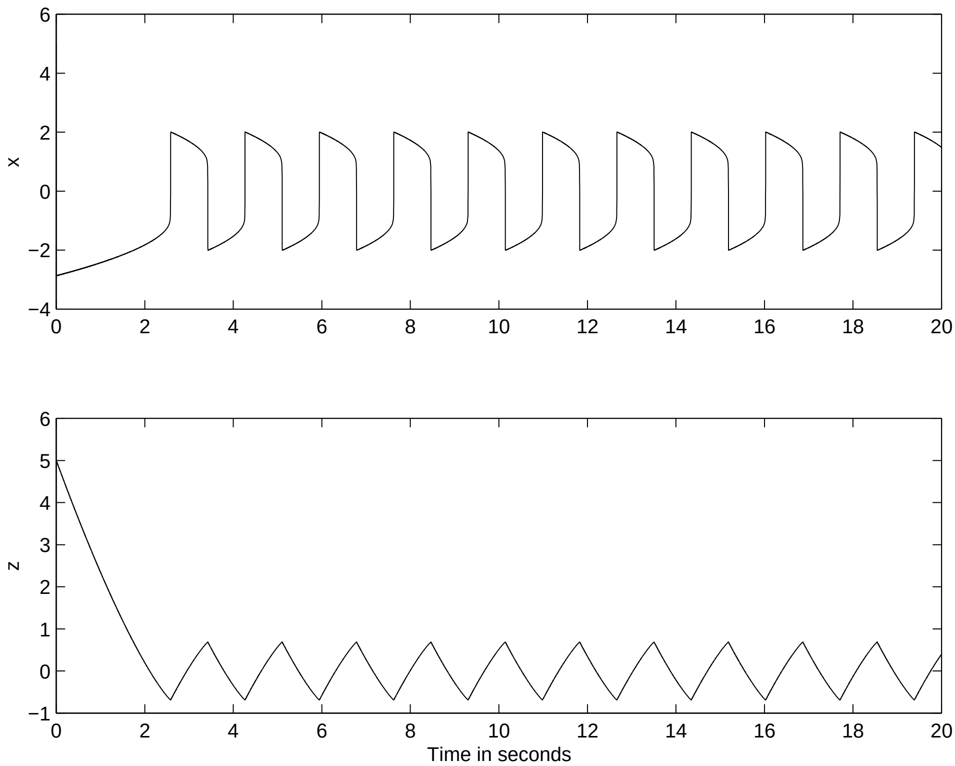


Figure 48: *State variables of a Van der Pol system,* $\varepsilon = 0.001$

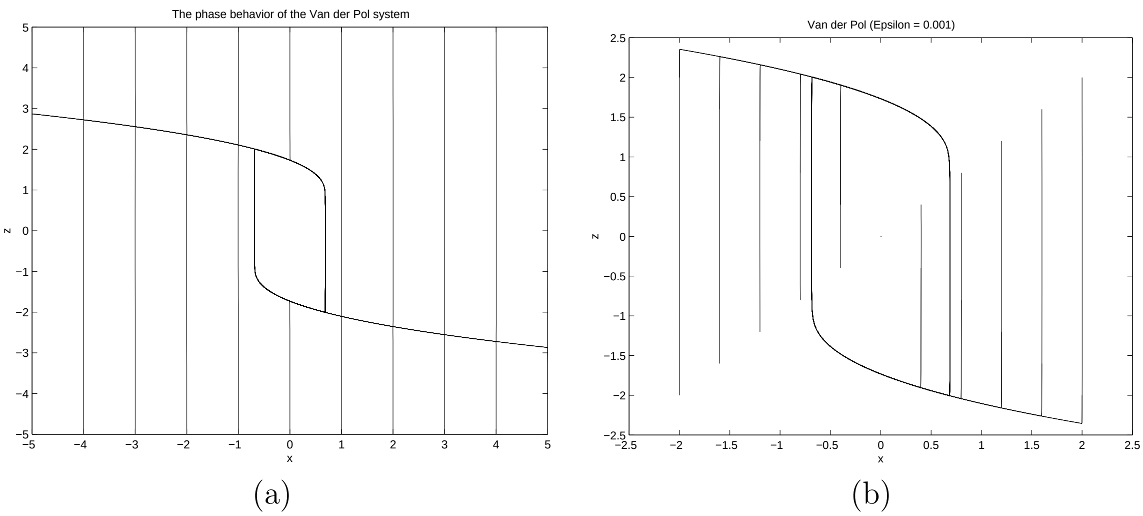


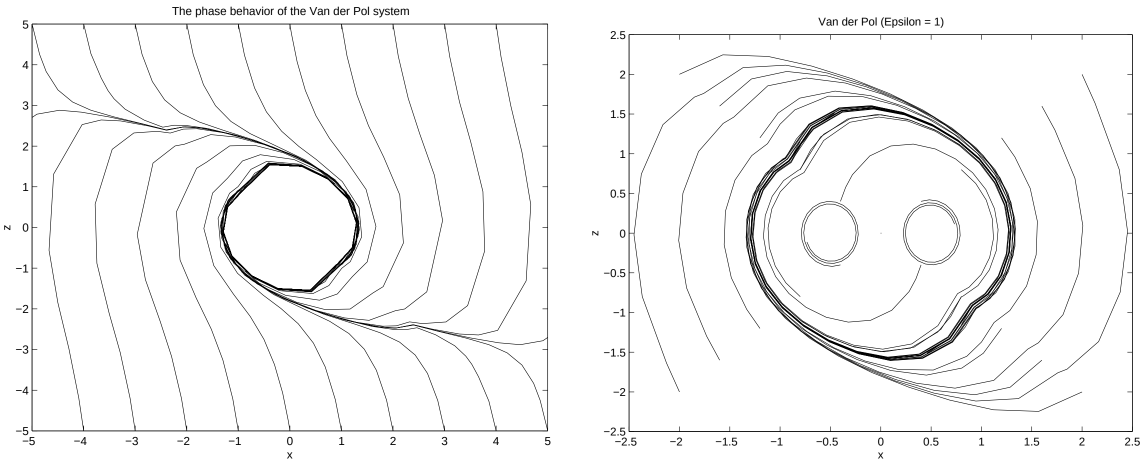
Figure 49: *Phase portrait of a Van der Pol system when* $\varepsilon = 0.001$.

Van der Pol with feedback of states and variable structure in the input

Next with a feed back

$$T = -x - z + \operatorname{sgn}(x + z)$$

the phase plots of the Van der Pol system are shown in Figure 50, 51 and 52 for $\varepsilon = 1$, $\varepsilon = 0.1$ and $\varepsilon = 0.001$ respectively.

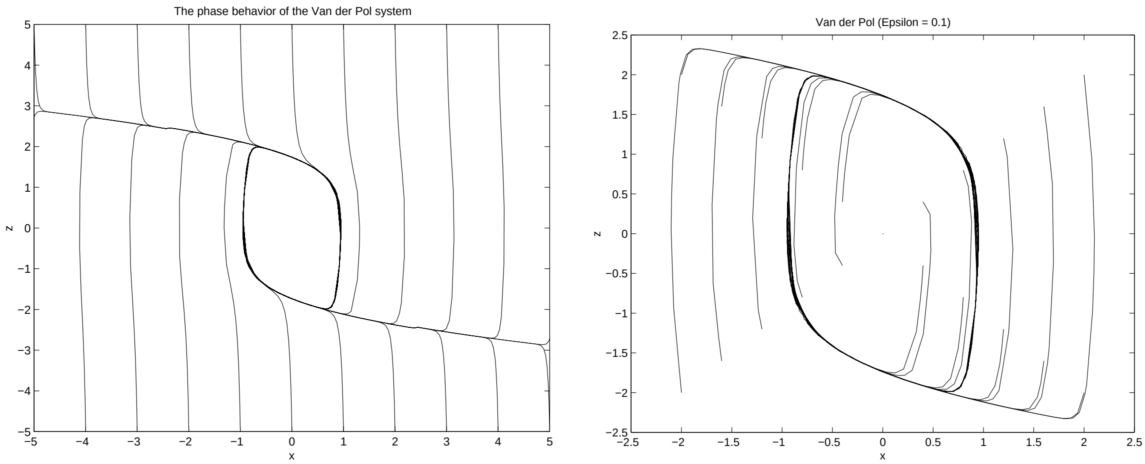


(a) (b)

Figure 50: *Phase portrait of a Van der Pol system with a variable structure control when* $\varepsilon = 1$.

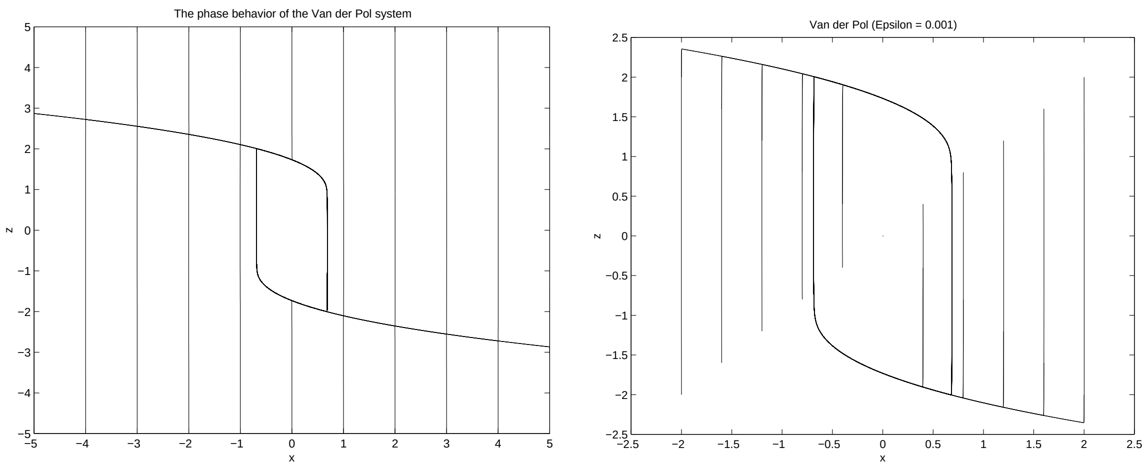
Discussion

The Van der Pol equation has a phase response which still remains the same characteristic, having a limit cycle of more or less the a similar shape and appearance, Figure 45, 47, 49 are similar to Figure 50, 51, 52 respectively. The shape and appearance of the state plane trajectory depends on ε.



(a) (b)

Figure 51: *Phase portrait of a Van der Pol system with VS control when* $\varepsilon = 0.1$.



(a) (b)

Figure 52: *Phase portrait of a Van der Pol system when* $\varepsilon = 0.001$.

A second order system

10^{th} **April 1998**
Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu \qquad (23) \\ y &= Cx + Du \qquad (24) \end{aligned}$$

with

$$A = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

and $D = 0$.

Some knowledge about this system

This system has got two poles at -1 ± 1 and one zero at -2. Figure 53 shows a root locus plot of this system. The only equilibrium point of the system is located at the origin because at the equilibrium point $Ax = 0$ which implies $x_1 = 0$ and $x_2 = 0$. The corresponding eigenvalues of this system are

$$-1 \pm \jmath 1.7321,$$

and thus the origin is a stable focus for the trajectories.

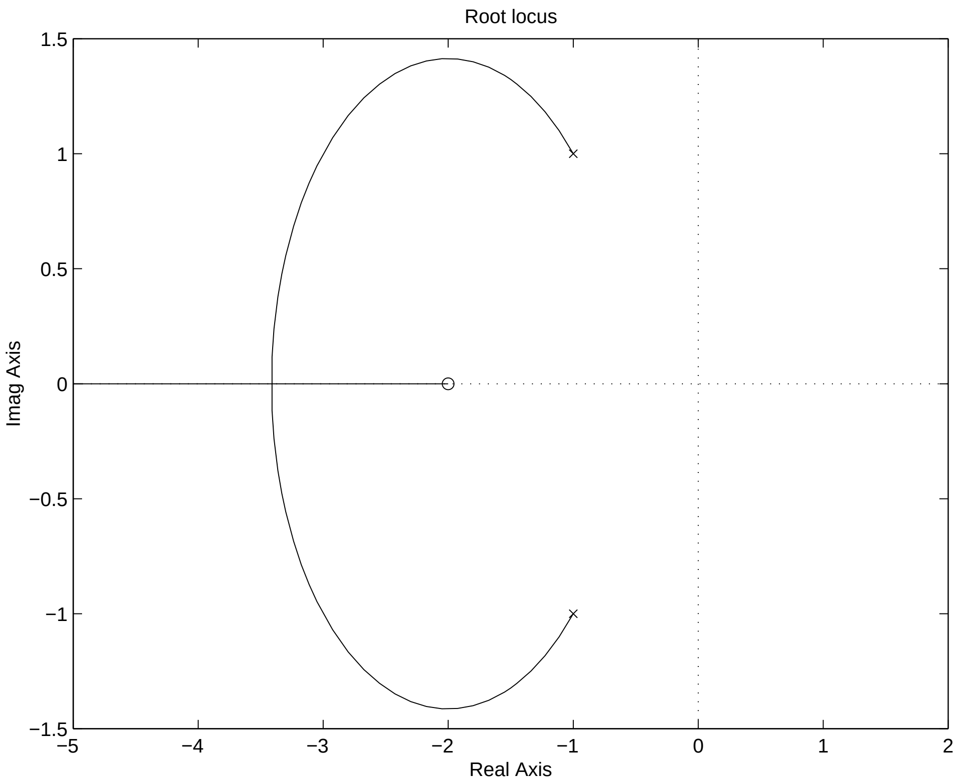


Figure 53: *Root locus of the two dimensional system.*

Sample response of the system with feedback of state variables and a signum function in the input

Figure 54 shows state diagrams of this system with various types of state feedback together with a sign function. The inputs used to simulate Figure 54 are

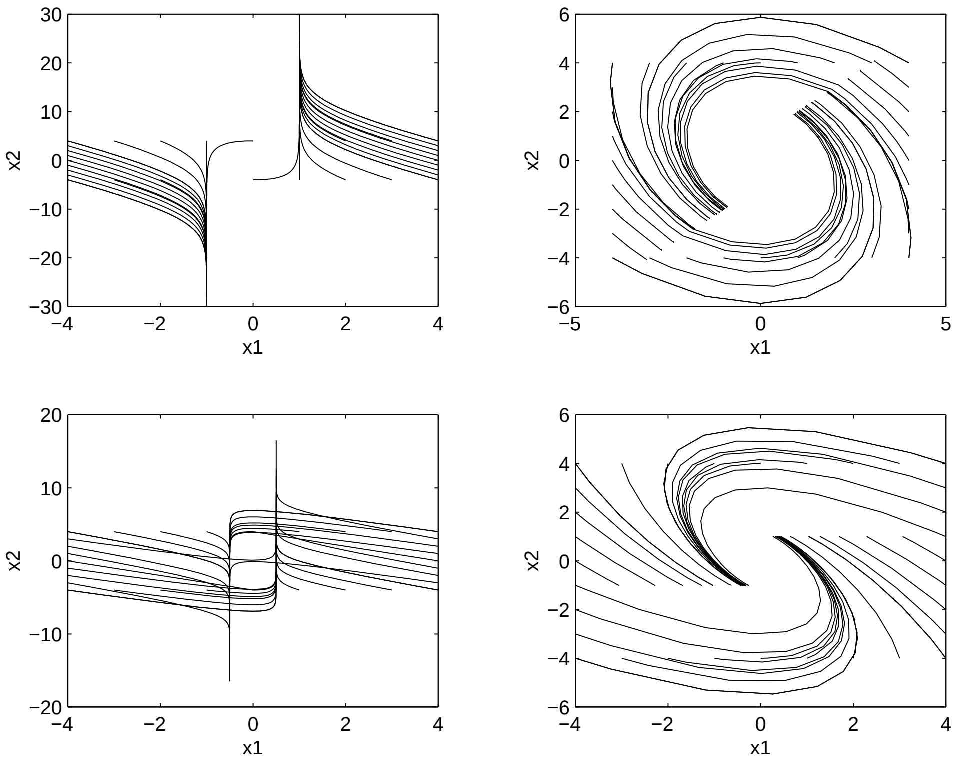
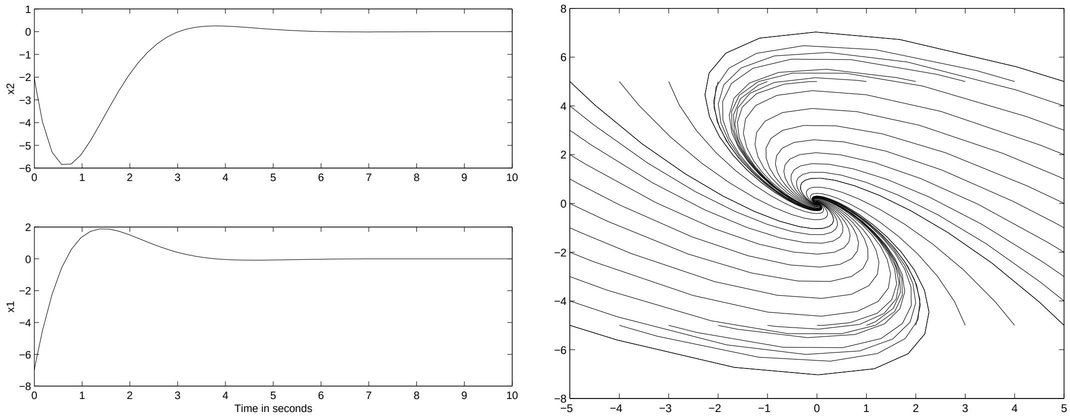


Figure 54: *From top down and from left to right, state diagrams of the system with a sign function together with a feedback of the two state variables, feed back of x_1, feed back of x_2 and feedback of neither one of the two state variables.*

$$u = Fx + \text{sgn}(CAx + CBu) = Fx + \text{sgn}\left(-x_2 + u\right),$$

where $F = [1\ \ 1], [1\ \ 0], [0\ \ 1]$ and $[0\ \ 0]$ from top left first. In simulating to obtain Figure 54 algebraic loops and discontinuities were encountered. The simulation was stopped manually periodically to be able to obtain the result.

The state variables and the state plane of Equation 23 are shown in Figure 55.



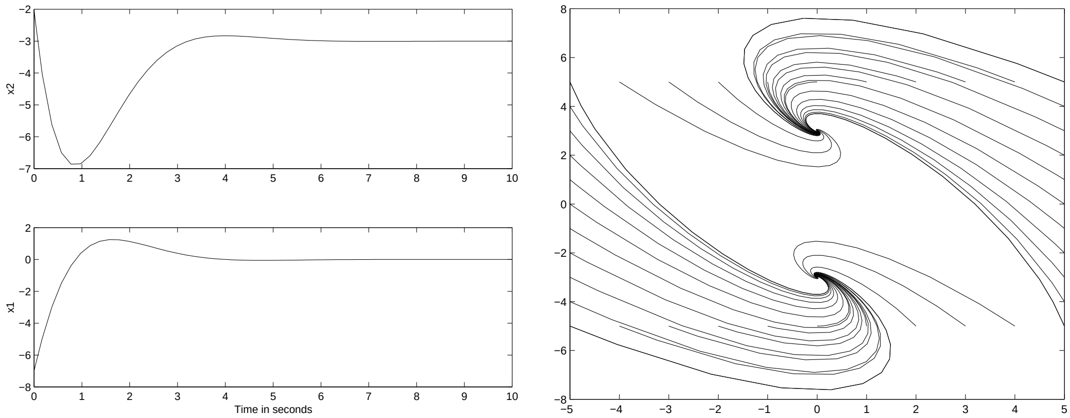
(a) (b)

Figure 55: *(a) The state variables* ($t_{simulation}$ = 10 sec, t_{CPU} = 0.0100 sec, *Initial conditions are* $(x_1, x_2) = (-7, -2)$*), and (b) the state plane* ($t_{simulation}$ = 10 sec, $t_{CPU,average}$ = 0.0077 sec, *Initial conditions are* $(x_1, x_2) \in \{-10, 10\} \times \{-10, 10\}$*)*

Let

$$u = 3\ \mathrm{sgn}(\mathrm{x}_1 + \mathrm{x}_2)$$

be the input of the system and the result as shown in Figure 56 was obtained.



(a) (b)

Figure 56: *(a) The state variables ($t_{simulation}$ = 10 sec, t_{CPU} = 0.0100 sec, Initial conditions are $(x_1, x_2) = (-7, -2)$), and (b) the state plane ($t_{simulation}$ = 10 sec, $t_{CPU,average}$ = 0.0102 sec, Initial conditions are $(x_1, x_2) \in \{-10, 10\} \times \{-10, 10\}$), $u = 3\ \mathrm{sgn}(\mathrm{x}_1 + \mathrm{x}_2)$*

Discussion

Hyperplanes can be seen in Figure 54, though they might be somewhat strange in appearance since the hyperplane here is

$$s(x) = -x_2 + u,$$

where u is the input. Figure 55 shows that the system trajectory has a character of a stable focus with an equilibrium point at the origin. Figure 56 shows a division of the focus into two foci, system trajectory is then moving toward one of these two stable foci (which one it goes to depends on the initial starting point of the trajectory.)

Simple pendulum

Next consider a nonlinear problem of a simple pendulum. [Kha96] (page 5) Let l be the length of the rod, m the mass of the bob and θ the angle subtended by the vertical axis through the pivot point and the rod . Assume that the rod is rigid and has zero mass. Let g be the gravitational constant and k be the coefficient on the bob moving through air the value of which is proportional to the speed of the bob. The equation of motion following Newton's second law of motion is then

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta} \tag{25}$$

The state equations of the system is

$$\begin{aligned}\dot{x_1} &= x_2 && (26)\\ \dot{x_2} &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \frac{T}{ml^2}, && (27)\end{aligned}$$

providing $x_1 = \theta$ and $x_2 = \dot{\theta}$.

Some knowledge about the system

Let $\dot{x_1} = \dot{x_2} = 0$ and solve the equations for the equilibrium points. We have

$$x_2 = 0 \quad \text{and} \quad \sin(x_1) = 0 \tag{28}$$

and therefore the equilibrium points are

$$(n\pi, 0), \quad n = 0, \pm 1, \pm 2, \ldots. \tag{29}$$

The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

and the Jacobian's estimated at the equilibrum points are

$$A = \left.\begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}\right|_{x=(n\pi,0)}$$

which have the eigenvalues of

$$\frac{1}{2ml}\left.\left[-\frac{l}{k} \pm \sqrt{l^2k^2 - 4m^2lg\cos x_1}\right]\right|_{x_1=n\pi}, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{30}$$

As an example, let

$$g = -9.8, \quad l = 1, \quad k = .45, \quad m = .5$$

and A becomes

$$A = \begin{bmatrix} 0 & 1 \\ \frac{49}{5}\cos x_1 & -\frac{9}{10} \end{bmatrix}$$

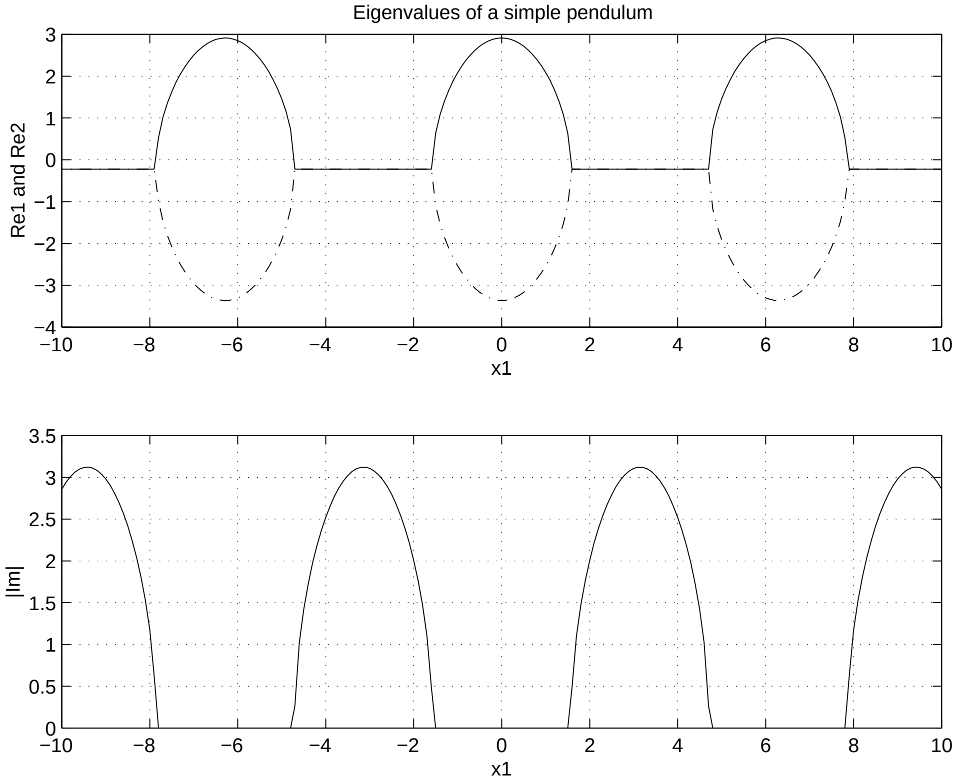


Figure 57: *Eigenvalues of a simple pendulum (it's real and imaginary components)*

which has

$$-\frac{9}{20} \pm \frac{1}{20}\sqrt{81 + 3920\cos x_1}$$

as eigenvalues. The real and imaginary components of the eigenvalues are shown in Figure 57.

From Equation 30 the term inside the square root is greater than zero when

$$\cos x_1 < \left|\frac{lk^2}{4m^2 g}\right| = |K|, \tag{31}$$

here only the absolute value is necessary because in real systems

$$l, k, m > 0,$$

but g can be either < 0 or > 0 depending on the convention used. According to Equation 31 there are two possibilities, that is

1. *Equation 31 is sometime true and sometime not.* This is the case shown in Figure 57 where $|K| = 0.02$. Another example is the case where

   $$l = 2.1, \quad k = 1.67, \quad , g = -9.8, \quad m = 0.7, \quad (|K| = 0.3).$$

   Figure 58 combines this case with Figure 57.

2. *Equation 31 can never be true.* This is the case demonstrated in Figure 59 where

$$l = 1, \quad k = 2, \quad , g = -9.8, \quad m = 0.14, \quad (|K| = 5.2).$$

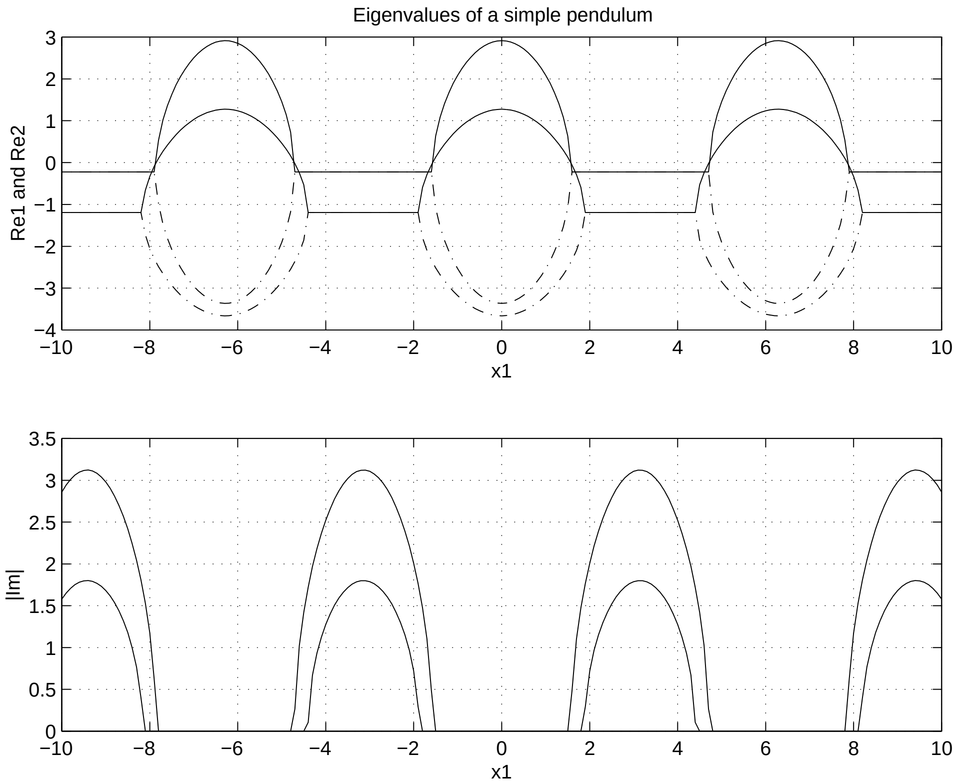


Figure 58: *Eigenvalues of a simple pendulum*

Depending on when and whether each of these two equations

$$\frac{1}{2ml}\left[-\frac{l}{k} \pm \sqrt{l^2k^2 - 4m^2lg\cos x_1}\right] \;<\; 0 \tag{32}$$

$$l^2k^2 - 4m^2lg\cos x_1 \;\geq\; 0 \tag{33}$$

is true, the characteristic of the trajectory can be determined. The characteristic of the trajectory can then be summarized as shown in Table 4.

Sample responses of a simple pendulum without an input

Figure 60 show the state variables of this system. Here $g = 9.8$, $l = 1$, $k = 1.5$ and $m = 3$ were used in simulation.

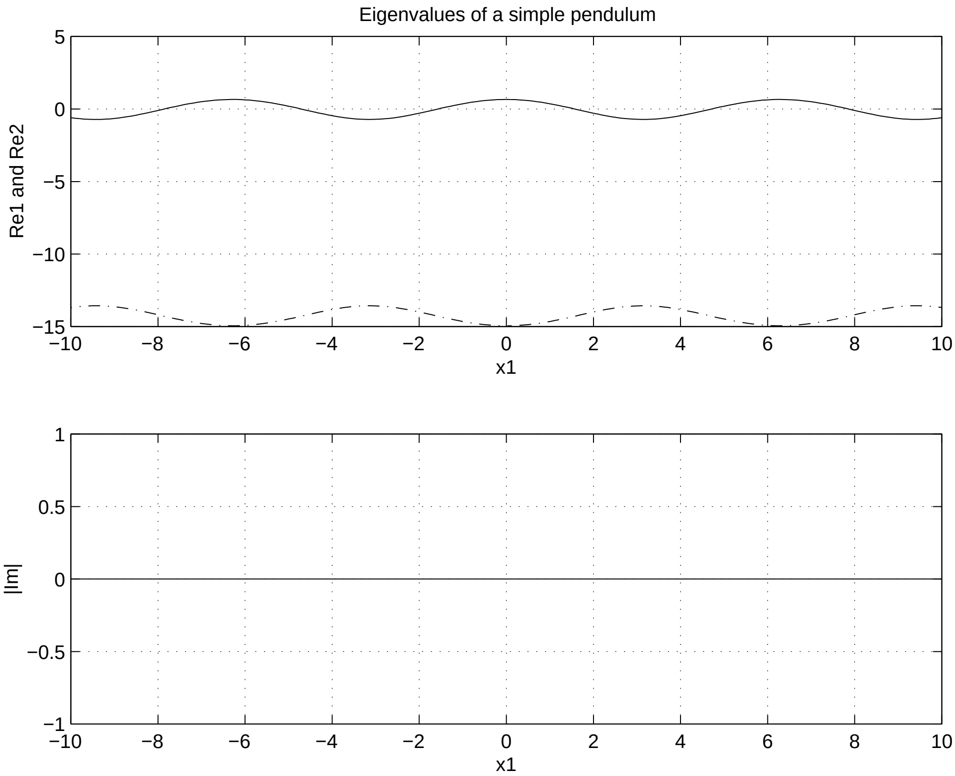


Figure 59: *Eigenvalues of a simple pendulum*

Value of x_1	Nature of the trajectory
$\left[\left(2n\pi - \cos^{-1}\frac{k^4}{4m^2lg}\right), \left(2n\pi + \cos^{-1}\frac{k^4}{4m^2lg}\right)\right]$	unstable node
$x_1 = 2n\pi \pm \cos^{-1}\frac{k^4}{4m^2lg}$	there are equilibrium hyperplanes
$\left(\left(2n\pi + \cos^{-1}\frac{k^4}{4m^2lg}\right), \left(2n\pi + \cos^{-1}\frac{lk^2}{4m^2g}\right)\right]$	stable node
$\left[\left(2n\pi - \cos^{-1}\frac{lk^2}{4m^2g}\right), \left(2n\pi - \cos^{-1}\frac{k^4}{4m^2lg}\right)\right)$	stable node
$\left(\left(2(n-1)\pi + \cos^{-1}\frac{lk^2}{4m^2g}\right), (2n\pi-)\cos^{-1}\frac{lk^2}{4m^2g}\right)$	stable focus

Table 4: *Characteristic of trajectory for the simple pendulum*

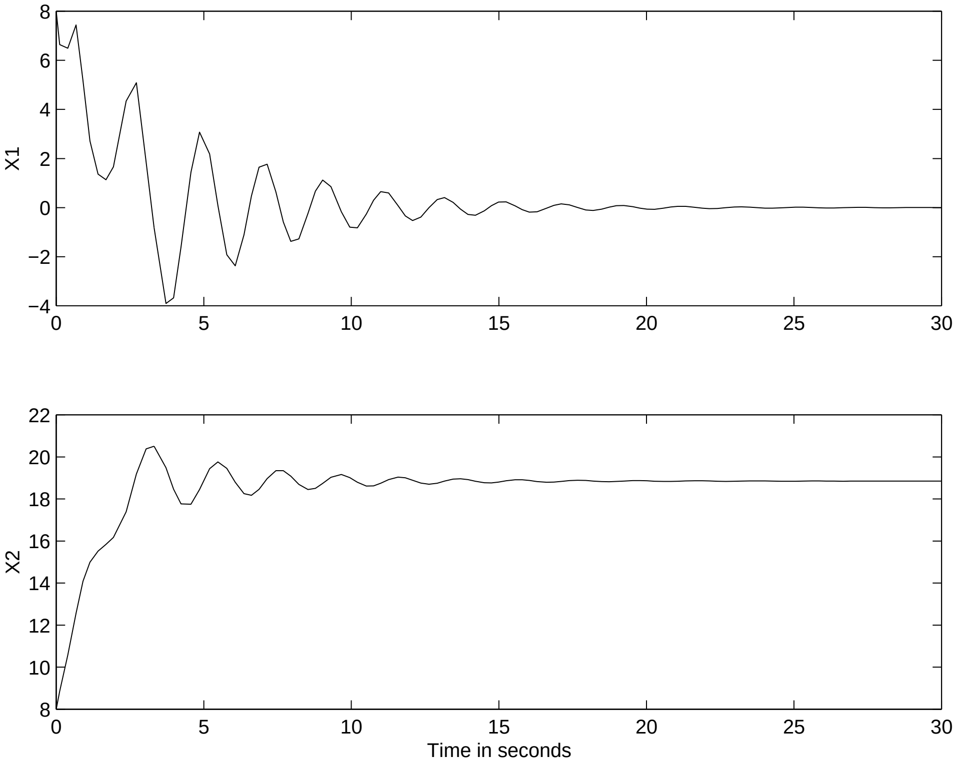


Figure 60: *State variables of a simple pendulum with initial conditions $x_1 = x_2 = 8$. No torque is applied.*

Then with initial conditions $(x_1, x_2) = (-10, 10)$, $(-6, 10)$, $(-2, 10)$, $(2, 10)$, $(6, 10)$, $(10, 10)$, $(-10, -10)$, $(-6, -10)$, $(-2, -10)$, $(2, -10)$, $(6, -10)$ and $(10, -10)$ draw the state diagram of the system as shown in Figure 61.

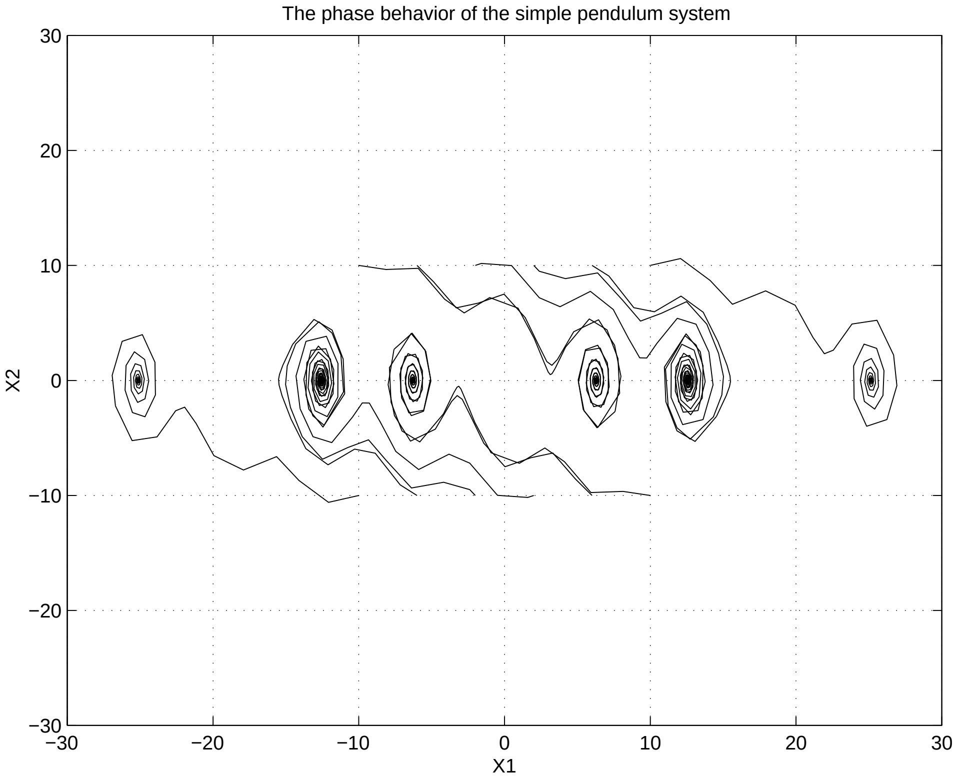


Figure 61: *State diagram of the simple pendulum with no input.*

A simple pendulum with various kinds of input

Figure 62 and Figure 63 are similar to Figure 60 and Figure 61 except that now there is an input $T = 1$.

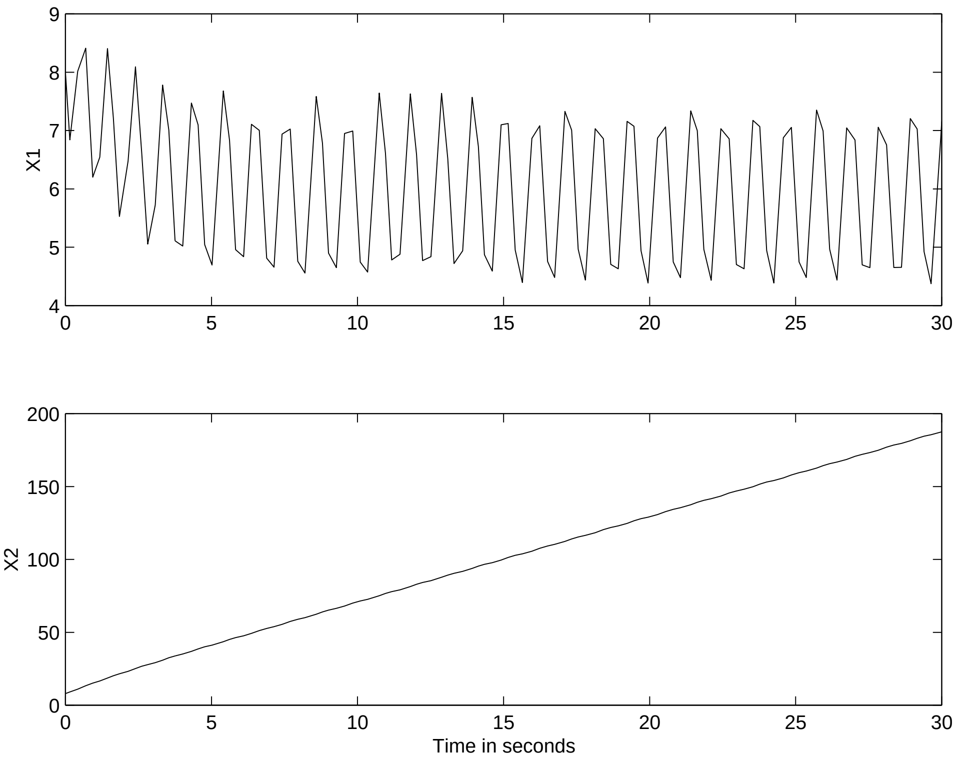


Figure 62: *State variables of a simple pendulum with initial conditions $x_1 = x_2 = 8$. Constant input $T = 1$. (Errata : the axis labels should be the other way around, ie the top figure should be labelled x_2 while the bottom one x_1)*

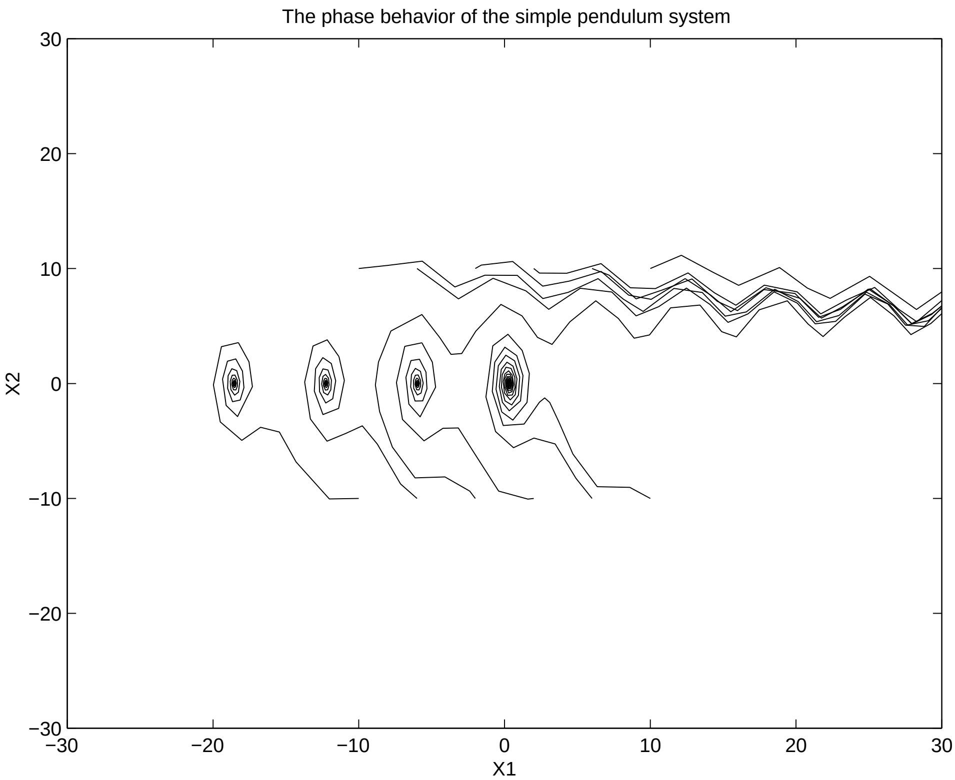


Figure 63: *State diagram of the simple pendulum with constant torque input.*

Figure 64 and Figure 65 are also similar to Figure 60 and Figure 61 and now the input is a sinusoidal torque of a unit amplitude (ie $T = \sin t$).

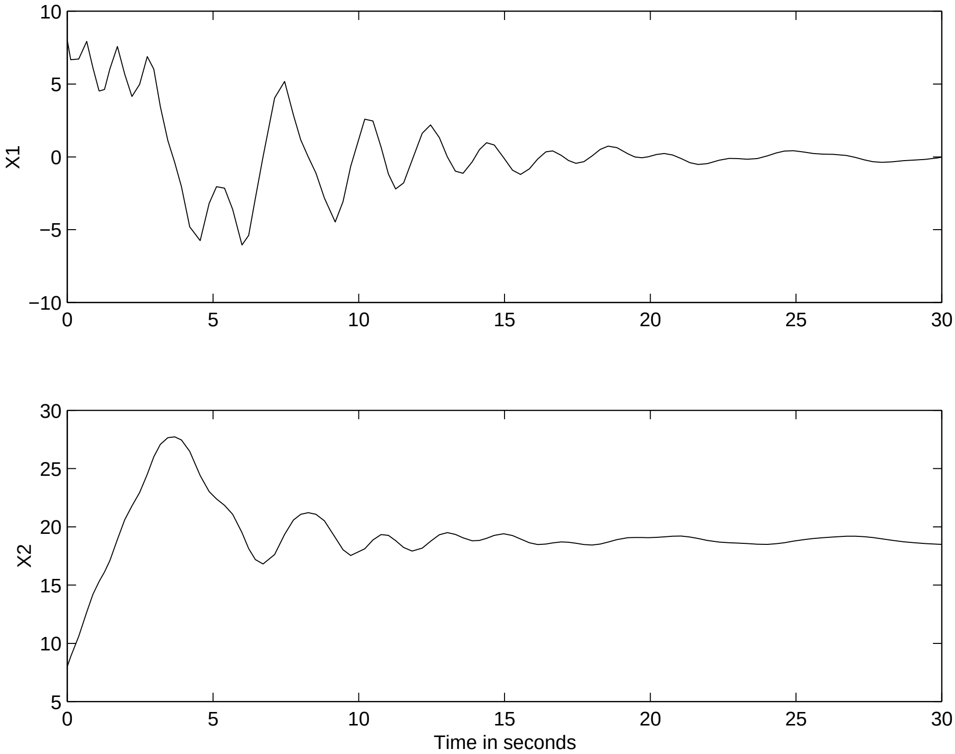


Figure 64: *State variables of a simple pendulum with initial conditions $x_1 = x_2 = 8$ and a sine input of a unit amplitude. (Errata : the axis labels should be the other way around, ie the top figure should be labelled x_2 while the bottom one x_1)*

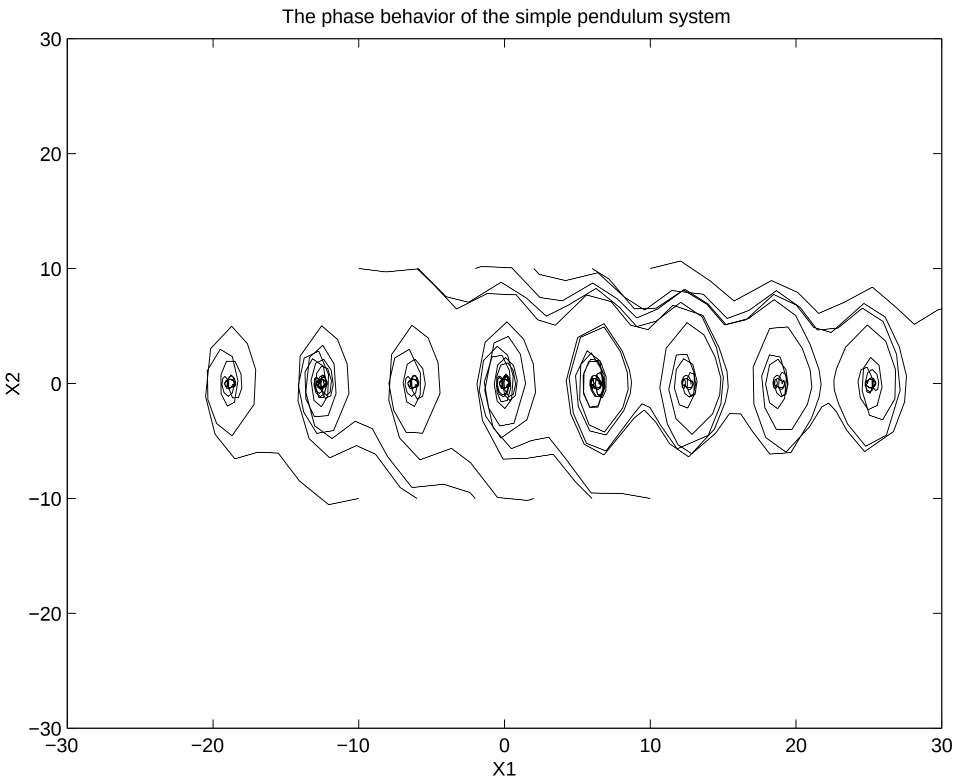


Figure 65: *State diagram of the simple pendulum with a sinusoidal input.*

Figure 66 and Figure 67 are also similar to Figure 60 and Figure 61 but with a feedback of state variables and a unit step input, ie

$$T = x_1 + x_2 + 1(t), \quad 1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

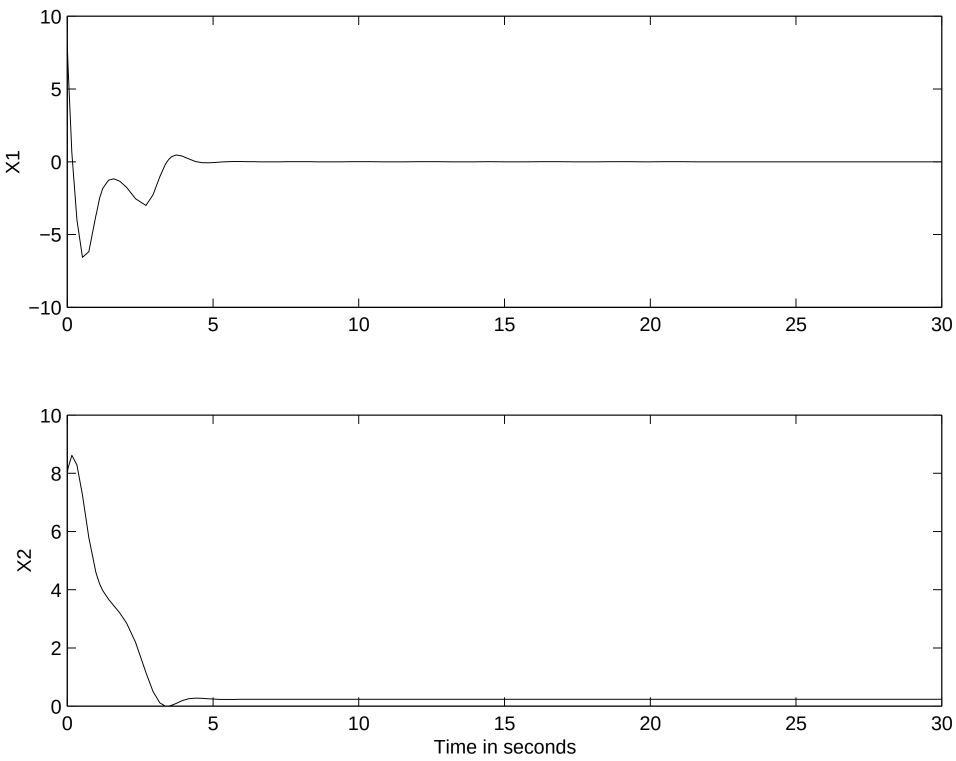


Figure 66: *State variables of a simple pendulum with initial conditions $x_1 = x_2 = 8$ and a unit step input and a feedback of state variables. (Errata : the axis labels should be the other way around, ie the top figure should be labelled x_2 while the bottom one x_1)*

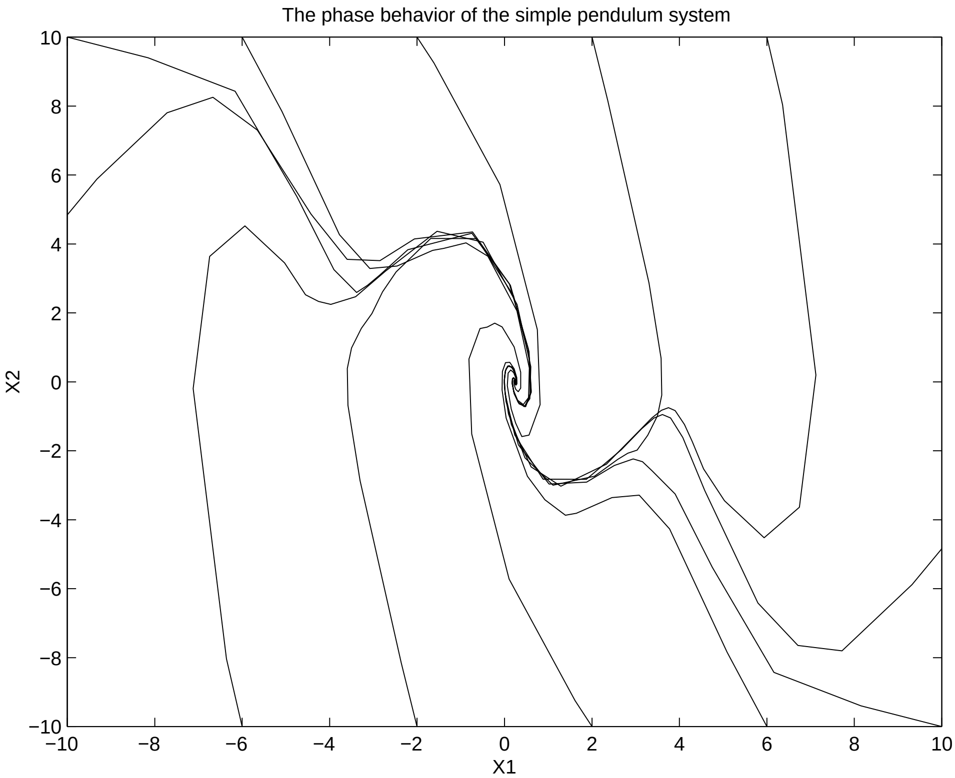


Figure 67: *State diagram of the simple pendulum with a unit step input and a feedback of state variables.*

Figure 68 and Figure 69 shows the results when

$$T = -x_1 - x_2 + \mathrm{sgn}(x_2).$$

The result shows a limit cycle.

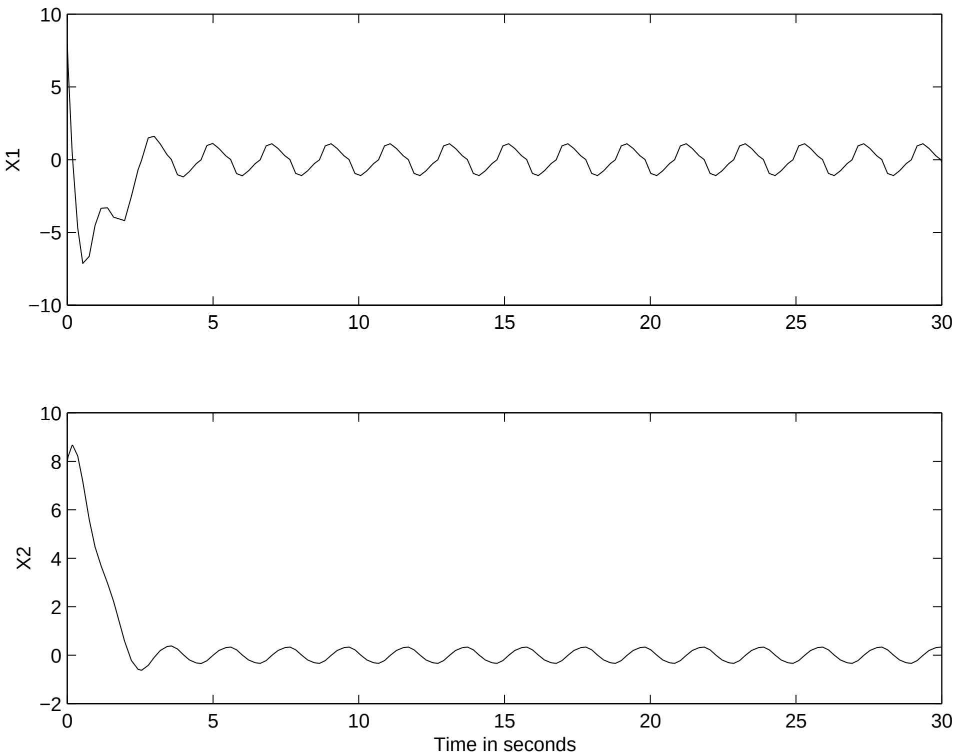


Figure 68: *State variables of a simple pendulum with initial conditions* $x_1 = x_2 = 8$ *and a sign function input and a feedback of state variables. (Errata : the axis labels should be the other way around, ie the top figure should be labelled* x_2 *while the bottom one* x_1*)*

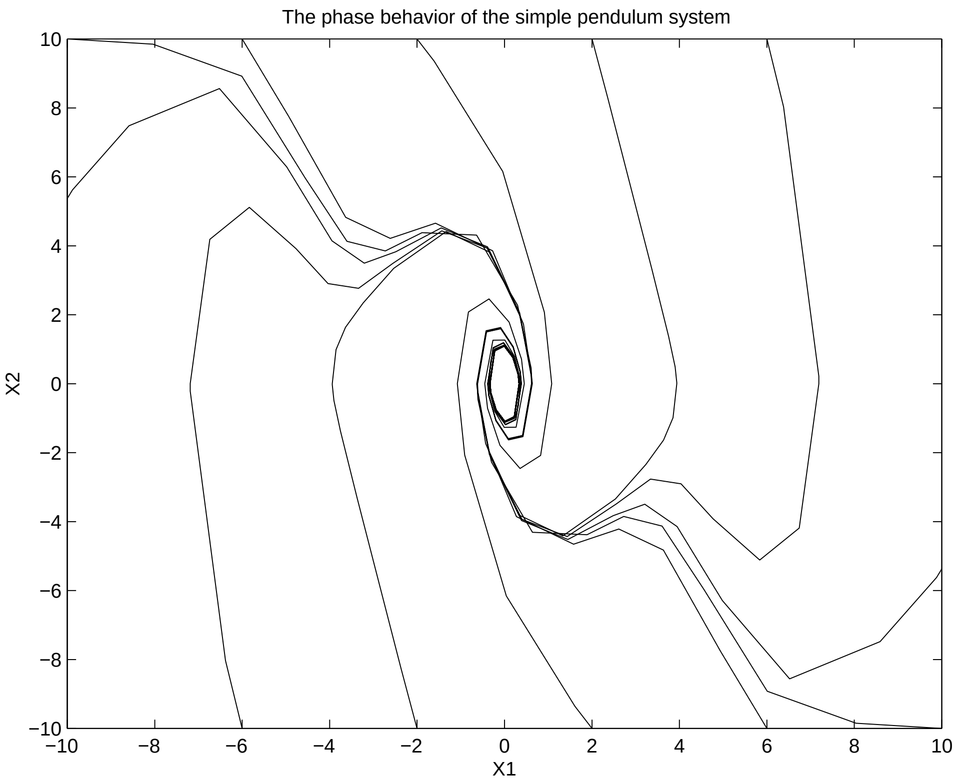


Figure 69: *State diagram of the simple pendulum with a feedback of state variablesand a sign function input.*

This time use a full state feed back together with a sign function of a hyperplane, that is

$$T = x_1 + x_2 + \mathrm{sgn}(x_1 + x_2).$$

The state variables as functions of time are shown in Figure 70 and the state diagram Figure 71. Figure 71 shows clearly sliding surfaces in the state plane.

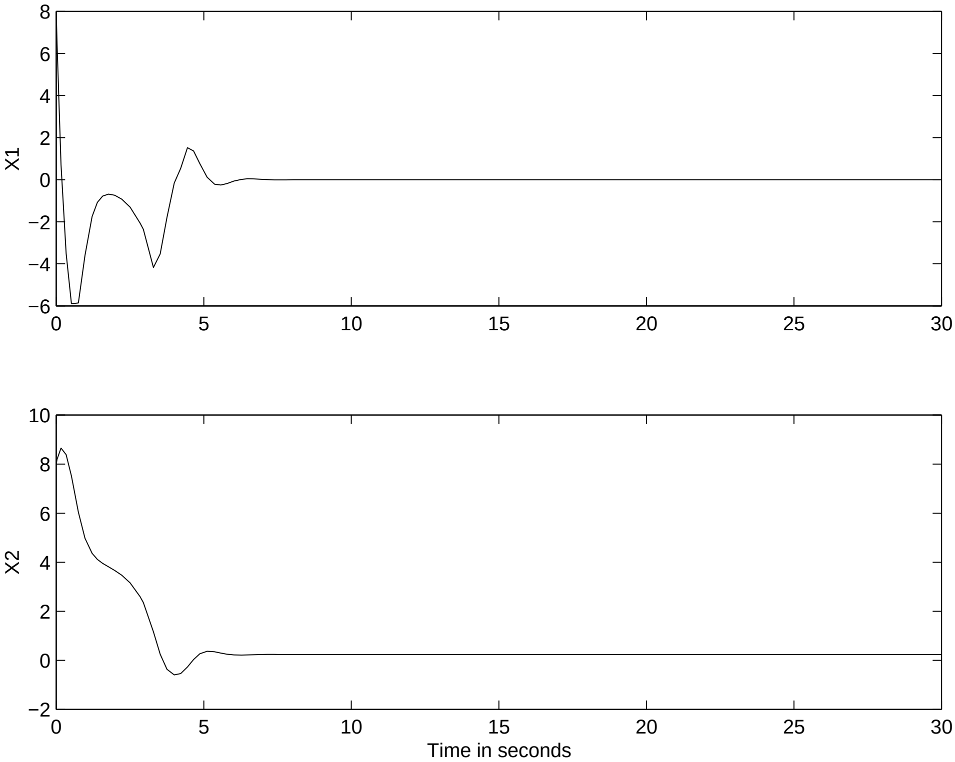


Figure 70: *State variables of a simple pendulum with initial conditions $x_1 = x_2 = 8$ and a hyperplane sign function input and a feedback of state variables. (Errata : the axis labels should be the other way around, ie the top figure should be labelled x_2 while the bottom one x_1)*

Discussion

With no input applied, Figure 60 shows that the pendulum swings and the goes to an equilibrium. Figure 61 shows many equilibrium points all of which is the same position in space (though having different angles θ) and having the angular velocity $\omega = \dot{theta} = 0$. When a constant input is applied the pendulum turns around and around in circle varying the angular velocity periodically (Figure 62 and Figure 63.)

When the pendulum is subjected to a sinusoidal input it still goes to equilibrium as Figure 64 and Figure 65 shows (though one might think that it should revolve on and on.)

Feeding back state variables and one other function, Figure 67 whos the trajectory which goes to the equilibrium point at the origin, Figure 69 has got a limit cycle, while in Figure 71 there

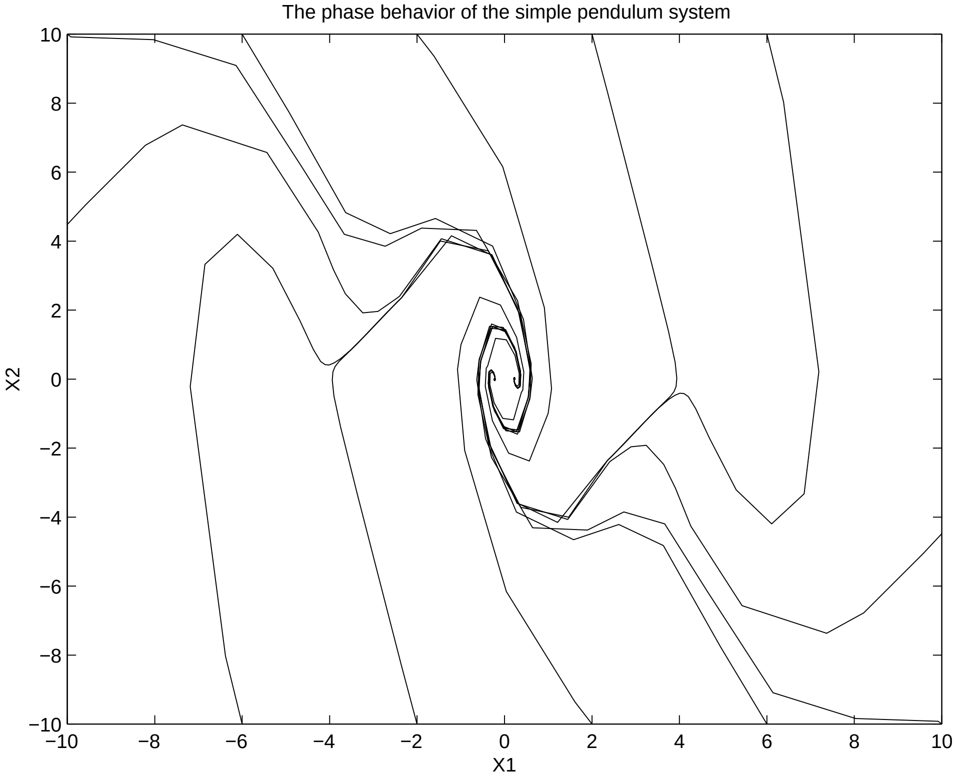


Figure 71: *State diagram of the simple pendulum with a feedback of state variables and a hyperplane sign function input.*

are two equilibrium points representing a pendulum pushed and stay on either side of the bottom resting position. Figure 66, 68 and 70 are time-domain plots of the state variables for the cases of the state planes shown in Figure 67, 69 and 71 respectively.

A fourth order system

2^{nd} April 1998

To study the effect which the parameter ε has on the poles and zeros location and the shape of the root locus of a system with the following transfer function was considered.

$$T(s) = \frac{2(s+5)(s^2+4s+29)}{(s+10)(s+1)(s^2+8s+20)}$$

This system have got poles at -10, -1, $-4+2i$ and $-4-2i$; zeros at -5, $-2+5i$ and $-2-5i$; and the gain $K=2$. The same system is then written in a state-space form.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

With

$$A = \begin{bmatrix} -8 & -5 & 1 & 1.25 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & -11 & -2.5 \\ 0 & 0 & 4 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 2.82843 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -2.82843 & 1.59099 & 0.70711 & 0.88388 \end{bmatrix}$$

and

$$D = 0.$$

The root locus plot of this system is the plot of the root of the equation $H(s) = 0$ where $H(s) = \frac{1+T(s)}{T(s)}$ and is shown in Figure 72.

A singularly perturbed system

Now consider the singular perturbation problem made by incorporating E into the system equation above.

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

Where $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon \end{bmatrix}$

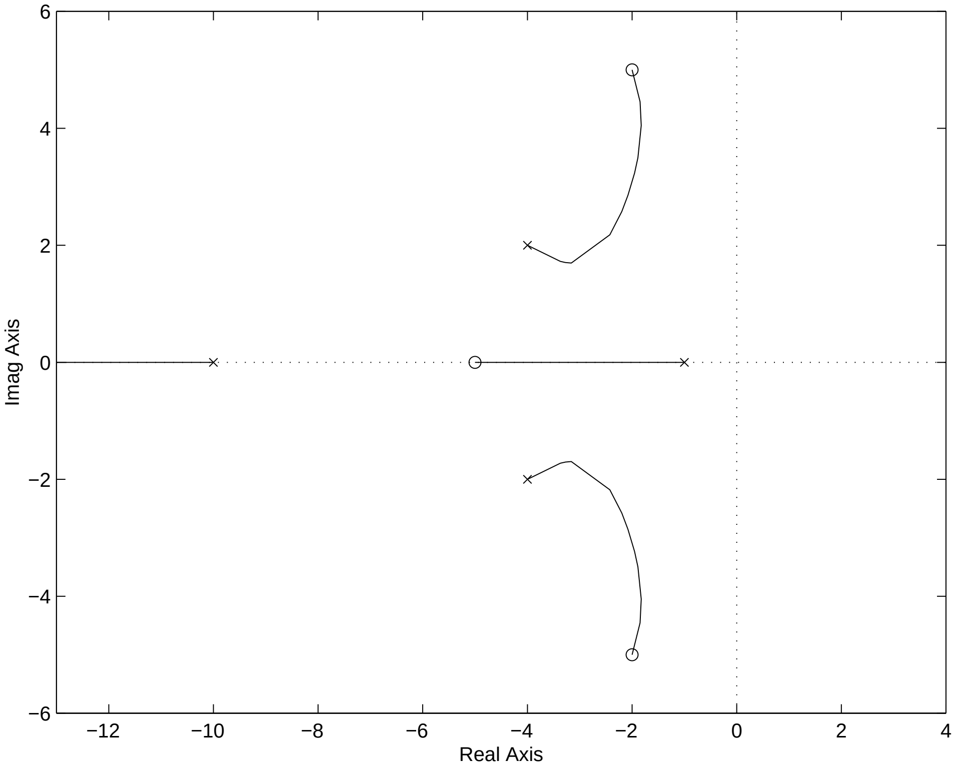


Figure 72: *Root locus of the original problem.*

Root locus of the system with various values of ε

Ther root locus plots of this system are shown in Figure 73, Figure 74, Figure 75 and Figure 76 for $\varepsilon = .5$, .1, .01 and .001 respectively. Figure 73 has got a transfer function

$$T(s) = \frac{2\ (s+10)\,(s^2+4\,s+29)}{(s^2+8\,s+20)\,(s^2+11\,s+20)},$$

Figure 74

$$T(s) = \frac{2\,s^3+108\,s^2+458\,s+2900}{(s^2+8\,s+20)\,(s^2+11\,s+100)},$$

Figure 75

$$T(s) = \frac{2\,s^3+1008\,s^2+4058\,s+29000}{(s^2+8\,s+20)\,(s^2+11\,s+1000)}$$

and Figure 76

$$T(s) = \frac{2\,s^3+10008\,s^2+40058\,s+(2.9\times 10^5)}{s^4+19\,s^3+10108\,s^2+80220\,s+(2\times 10^5)}.$$

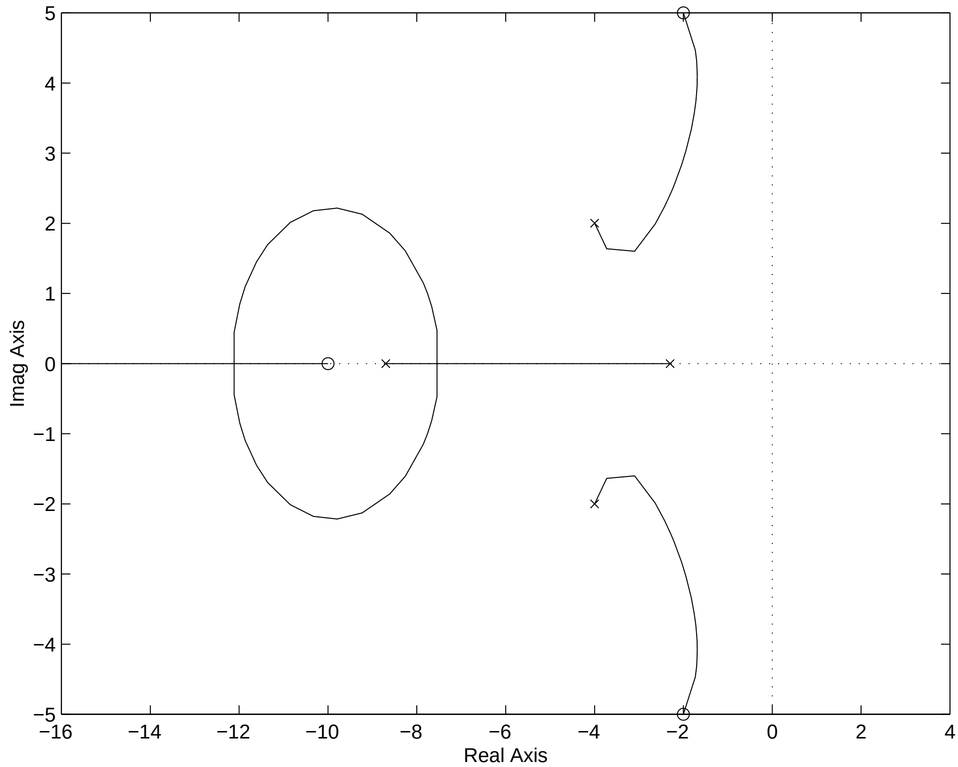


Figure 73: *Root locus when* $\varepsilon = .5$.

How ε affects the system characteristic

The value of ε which gives a pair of double poles on the real-axis was found by trials and errors to be $\varepsilon \approx 0.3306$, at which point the pole location was approximately -5.5 (actual values are approximately -5.46 and -5.54) and the corresponding transfer function was

$$T(s) = \frac{2s^3 + 38.25s^2 + 179s + 877.4}{s^4 + 19s^3 + 138.3s^2 + 462s + 605.1}.$$

The other two complex poles remain at the same place all the time at $-4 \pm 2\imath$.

The gain margin remains all the time at ∞ while the phase margin remains all the time at deg 126.9141 at the frequency of 0.9730 rad/sec. The Bode plot, singular value plot, Nyquist plot and Nichols plot are shown in Figure 77, Figure 78, Figure 79 and Figure 80 respectively.

As ε approaches zero

To observe the effect of ε on the system characteristic when $\varepsilon \to 0$ show Figure 77, Figure 78, Figure 79 and Figure 80 again for ε very small. The Bode plot, singular value plot, Nyquist plot and Nichols plot are shown again in Figure 77, Figure 78, Figure 79 and Figure 80 respectively.

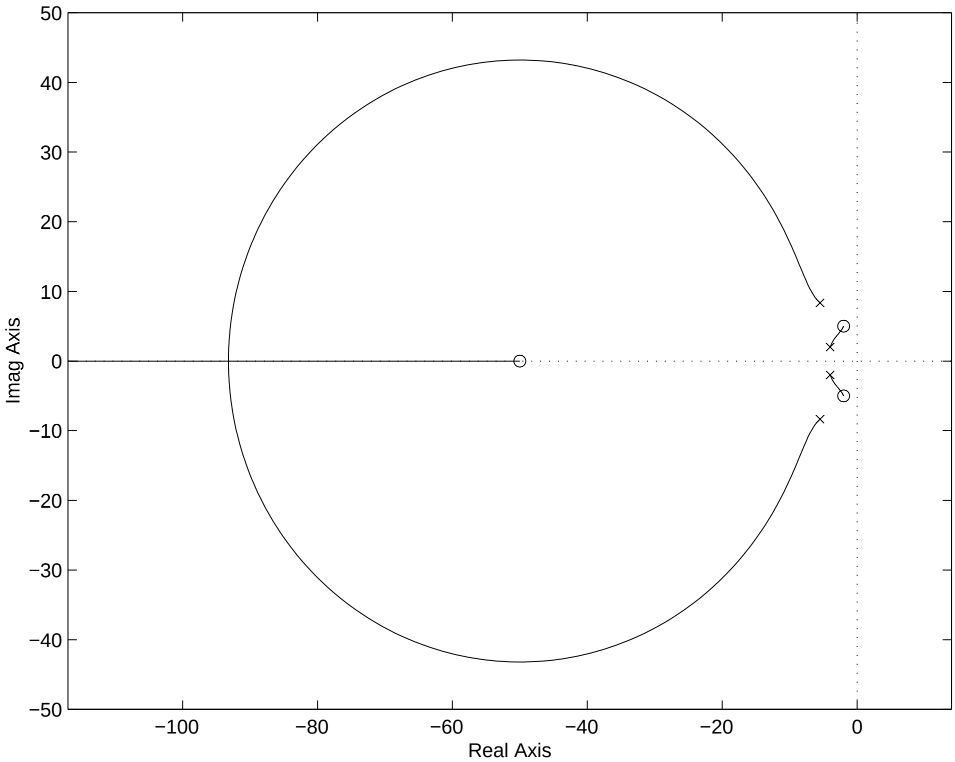


Figure 74: *Root locus when* $\varepsilon = .1$.

Some knowledge about this system

4^{th} **April 1998**

Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu \quad (34) \\ y &= Cx + Du \quad (35) \end{aligned}$$

with

$$A = \begin{bmatrix} -8 & -5 & 1 & 1.25 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & -11 & -2.5 \\ 0 & 0 & 4 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 2.82843 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -2.82843 & 1.59099 & 0.70711 & 0.88388 \end{bmatrix}$$

and

$$D = 0.$$

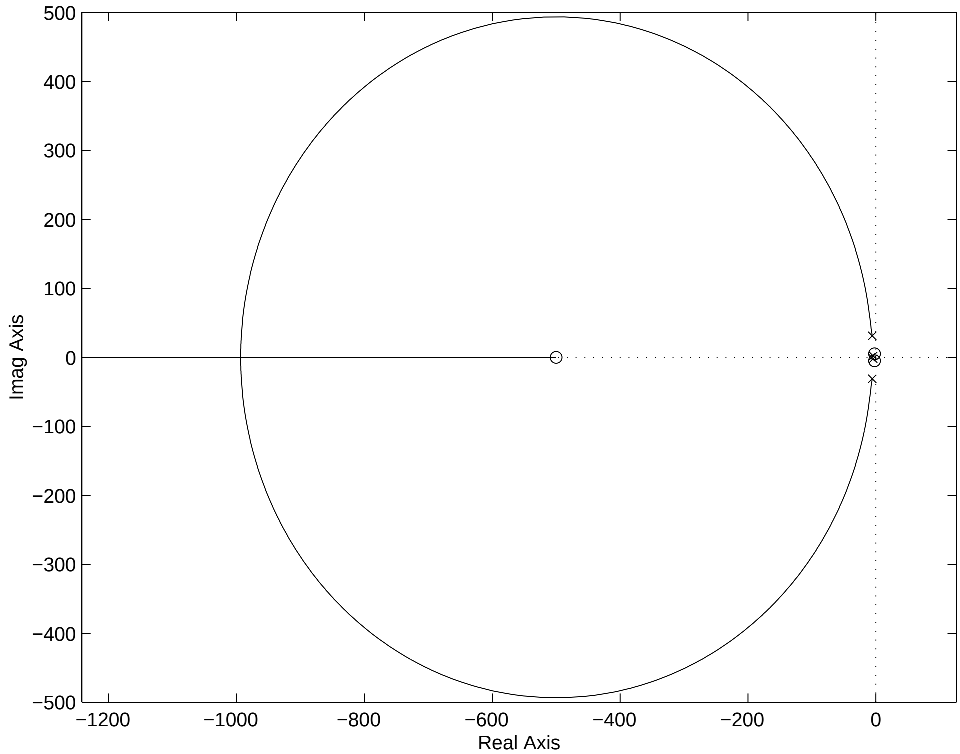


Figure 75: *Root locus when* $\varepsilon = .01$.

From the system equation the only equlibrium point is the origin. (because $Ax = 0$) The eigenvalues are

$$-4 \pm \jmath 2, \quad -1 \quad \text{and} \quad -10.$$

Response of the system with a feedback input

Design a controller with an input signal

$$u = Cx + u_i,$$

where $u_i = \text{sgn(CAx+CBu)}$ and sgn(f) is called a sign function that returns the sign of the function f. Next simulate this system ten steps for six times, each with a different initial condition $x(0)$. These $x(0)$ are $[2\ 1\ 13\ 14]^T$, $[-2\ \ -4\ \ -5\ \ -3]^T$, $[1\ 2\ \ -3\ \ -4]^T$, $[-3\ 1\ \ -2\ 5]^T$, $[-1\ 3\ 2\ \ -5]^T$ and $[2\ \ -5\ 4\ \ -3]^T$. Then draw state diagrams for some of the overall six parameter pairs.

Figure 85 shows four such diagrams. Notice that there is a distinct sliding hyperplane in each diagram.

With $x = [5\ 5\ 5\ 5]^T$, $[-5\ \ -5\ \ -5\ \ -5]^T$, $[5\ 5\ \ -5\ \ -5]^T$, $[-5\ \ -5\ 5\ 5]^T$, $[-5\ 5\ 5\ \ -5]^T$ and $[5\ \ -5\ \ -5\ 5]^T$ as initial conditions do a similar simulation with the system with a ε

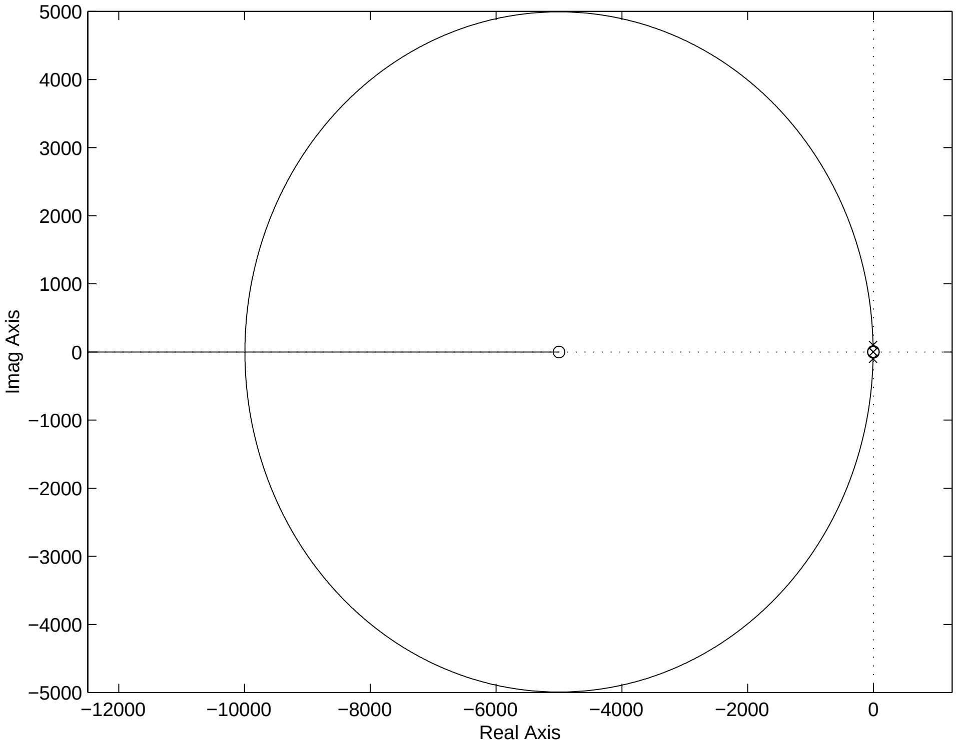


Figure 76: *Root locus when* $\varepsilon = .001$.

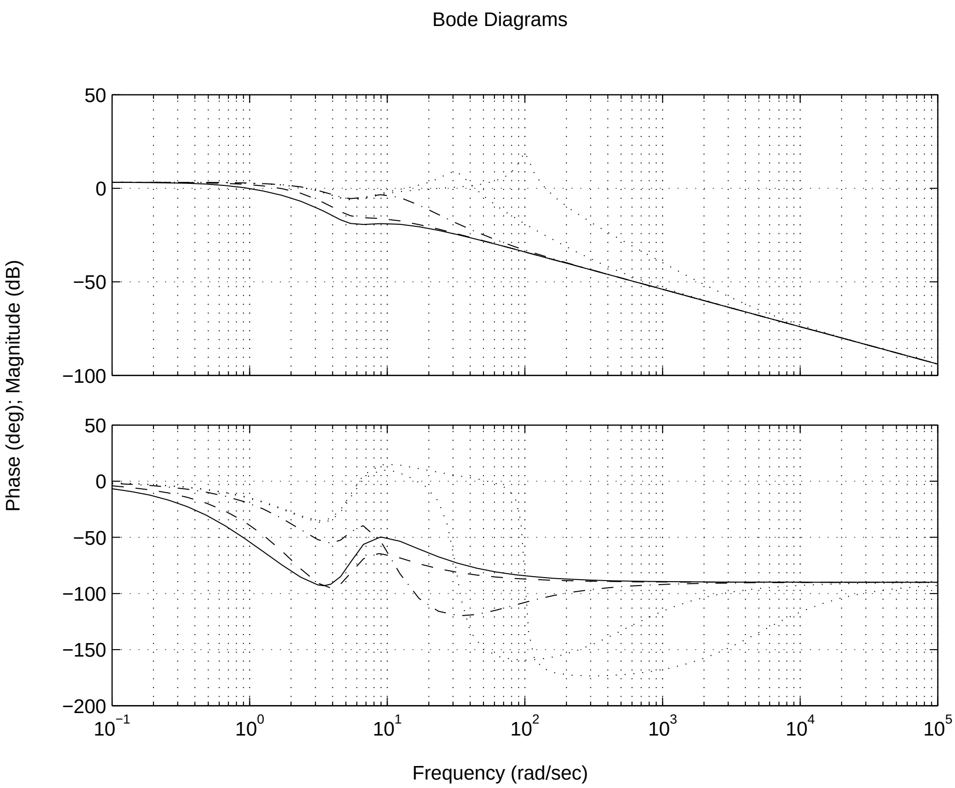


Figure 77: *Bode plot ($\varepsilon = 1$ as a solid line, $\varepsilon = 0.5$ a dashed line, $\varepsilon = 0.1$ a dashdot line, both $\varepsilon = 0.01$ and $\varepsilon = 0.001$ as dotted lines)*

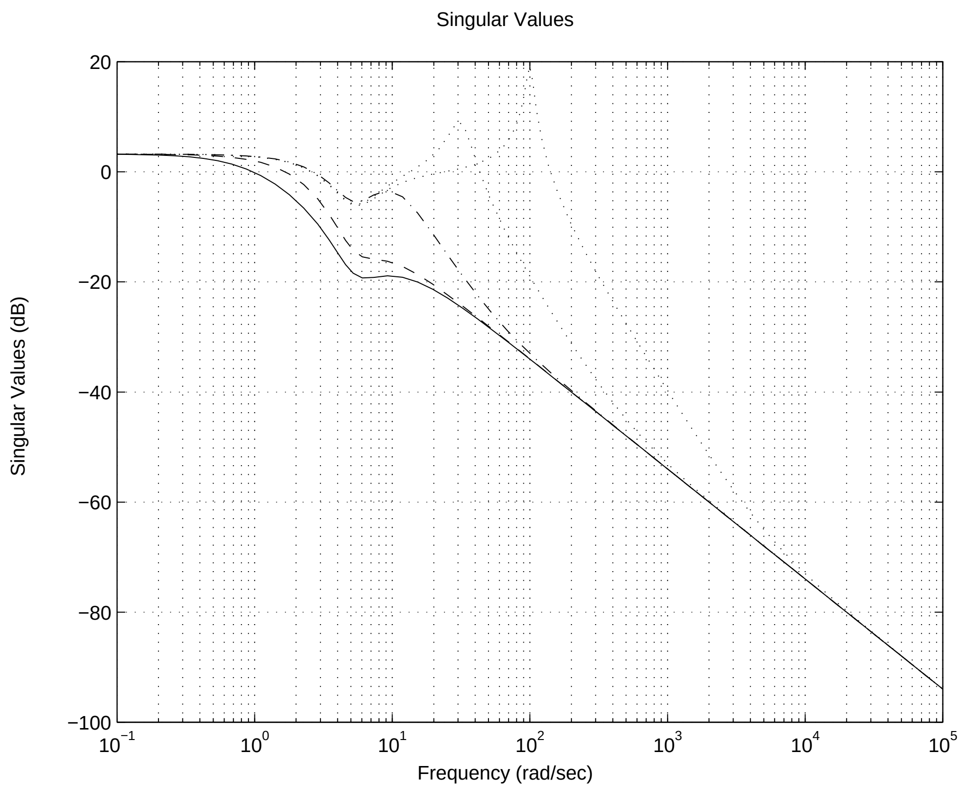


Figure 78: *Singular value plot ($\varepsilon = 1$ as a solid line, $\varepsilon = 0.5$ a dashed line, $\varepsilon = 0.1$ a dashdot line, both $\varepsilon = 0.01$ and $\varepsilon = 0.001$ as dotted lines)*

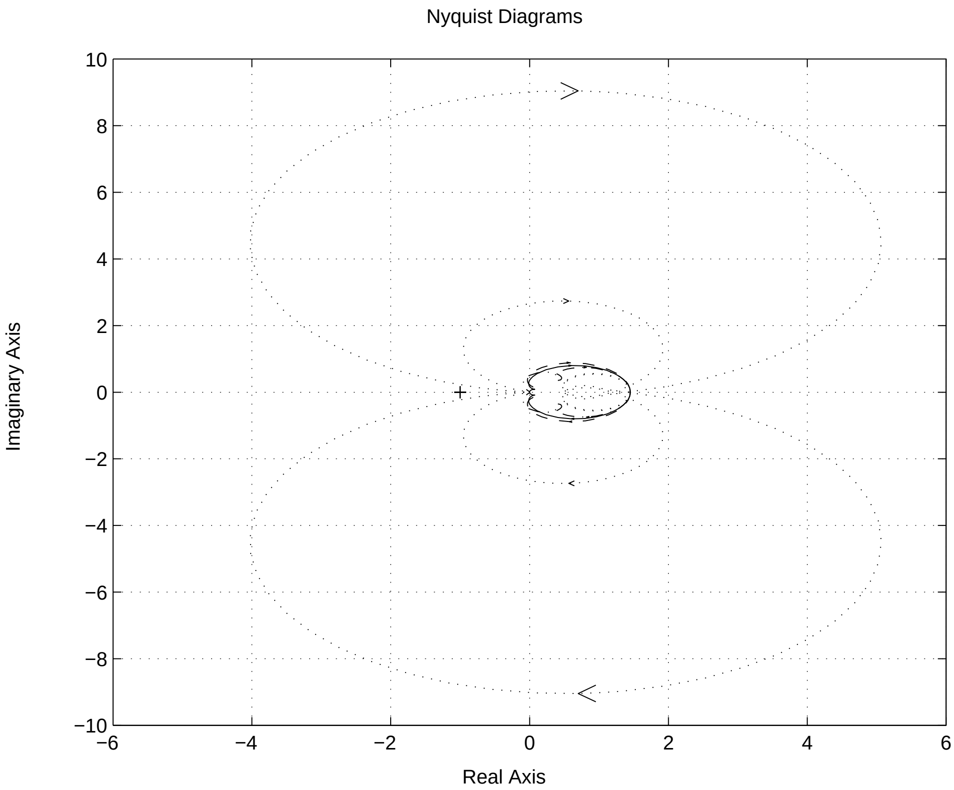


Figure 79: *Nyquist plot ($\varepsilon = 1$ as a solid line, $\varepsilon = 0.5$ a dashed line, $\varepsilon = 0.1$ a dashdot line, both $\varepsilon = 0.01$ and $\varepsilon = 0.001$ as dotted lines)*

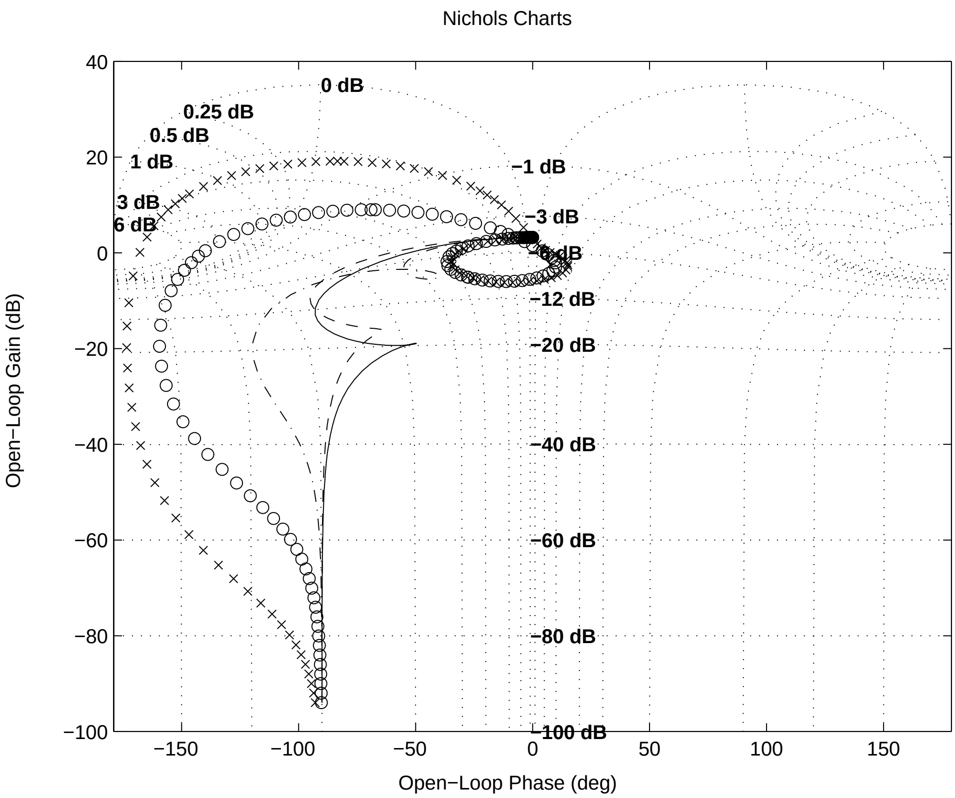


Figure 80: *Nichols plot ($\varepsilon = 1$ as a solid line, $\varepsilon = 0.5$ a dashed line, $\varepsilon = 0.1$ a dashdot line, $\varepsilon = 0.01$ circles and $\varepsilon = 0.001$ cross marks)*

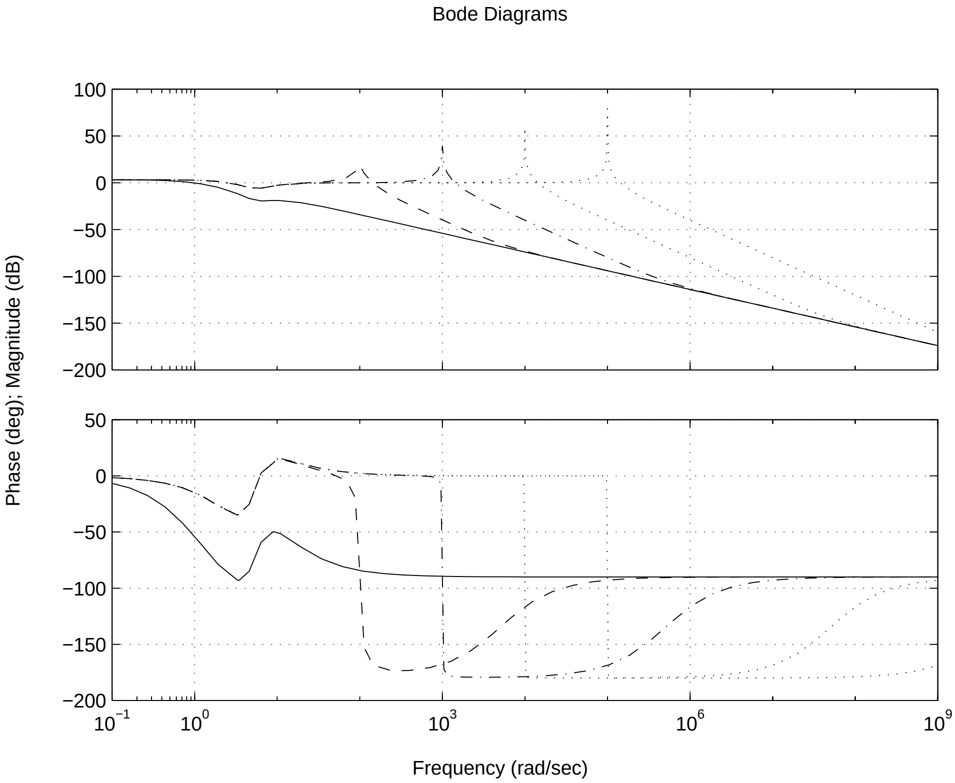


Figure 81: *Bode plot ($\varepsilon = 1$ as a solid line, $\varepsilon = 0.001$ a dashed line, $\varepsilon = 10^{-5}$ a dashdot line, both $\varepsilon = 10^{-7}$ and $\varepsilon = 10^{-9}$ as dotted lines)*

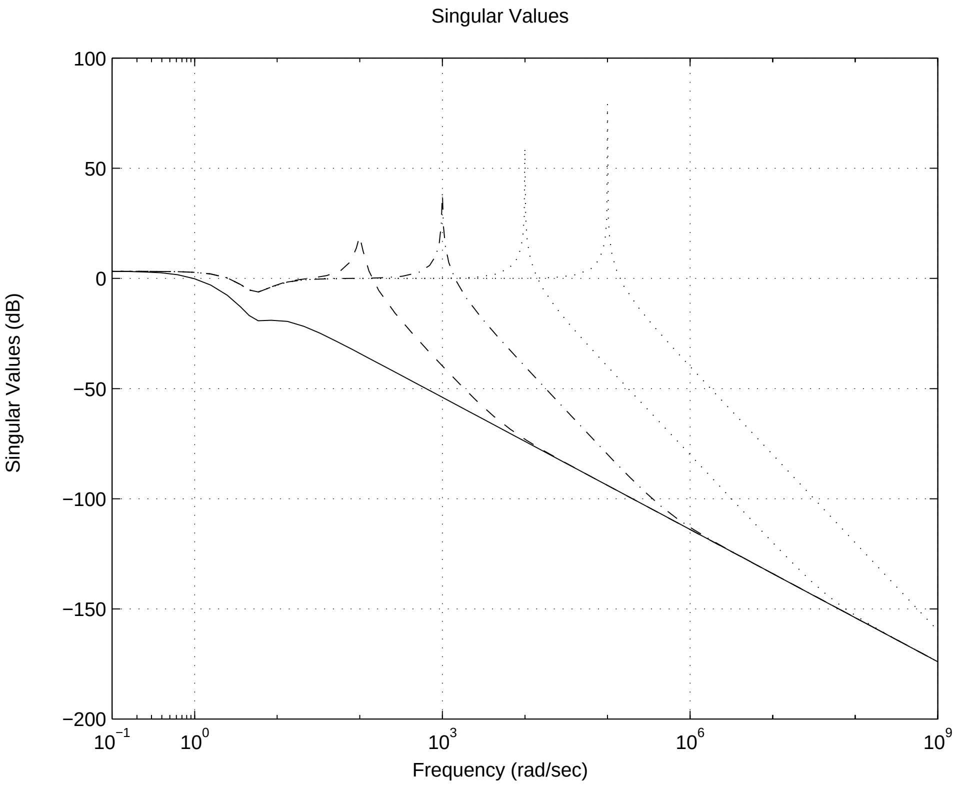


Figure 82: *Singular value plot ($\varepsilon = 1$ as a solid line, $\varepsilon = 0.001$ a dashed line, $\varepsilon = 10^{-5}$ a dashdot line, both $\varepsilon = 10^{-7}$ and $\varepsilon = 10^{-9}$ as dotted lines)*

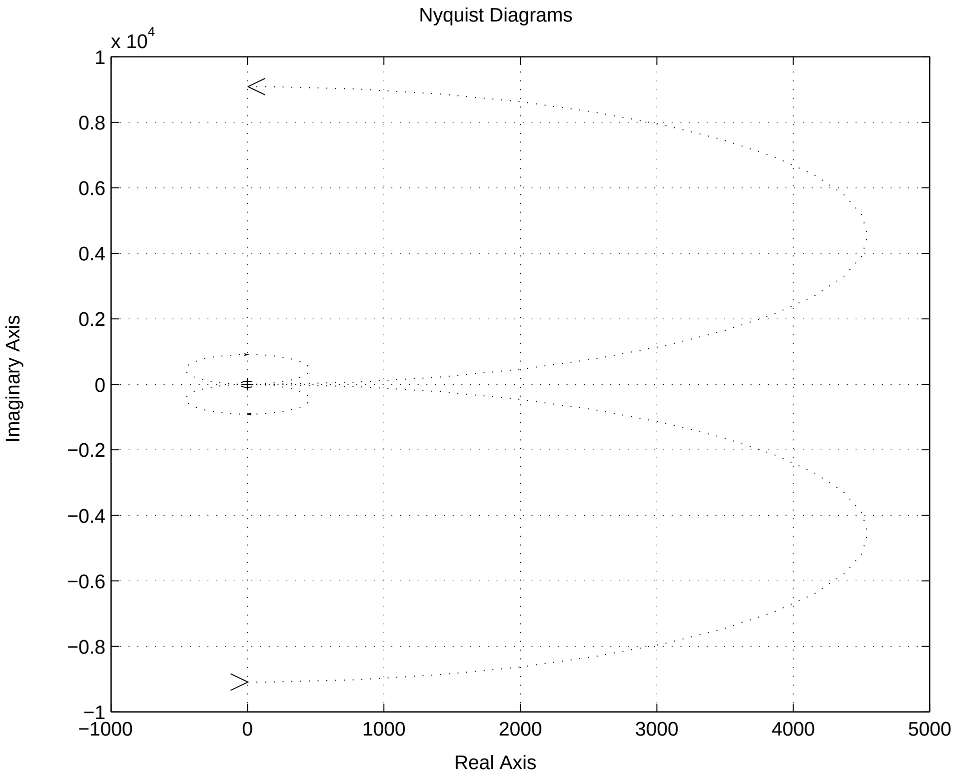


Figure 83: *Nyquist plot ($\varepsilon = 1$ as a solid line, $\varepsilon = 0.001$ a dashed line, $\varepsilon = 10^{-5}$ a dashdot line, $\varepsilon = 10^{-7}$ circles and $\varepsilon = 10^{-9}$ cross marks)*

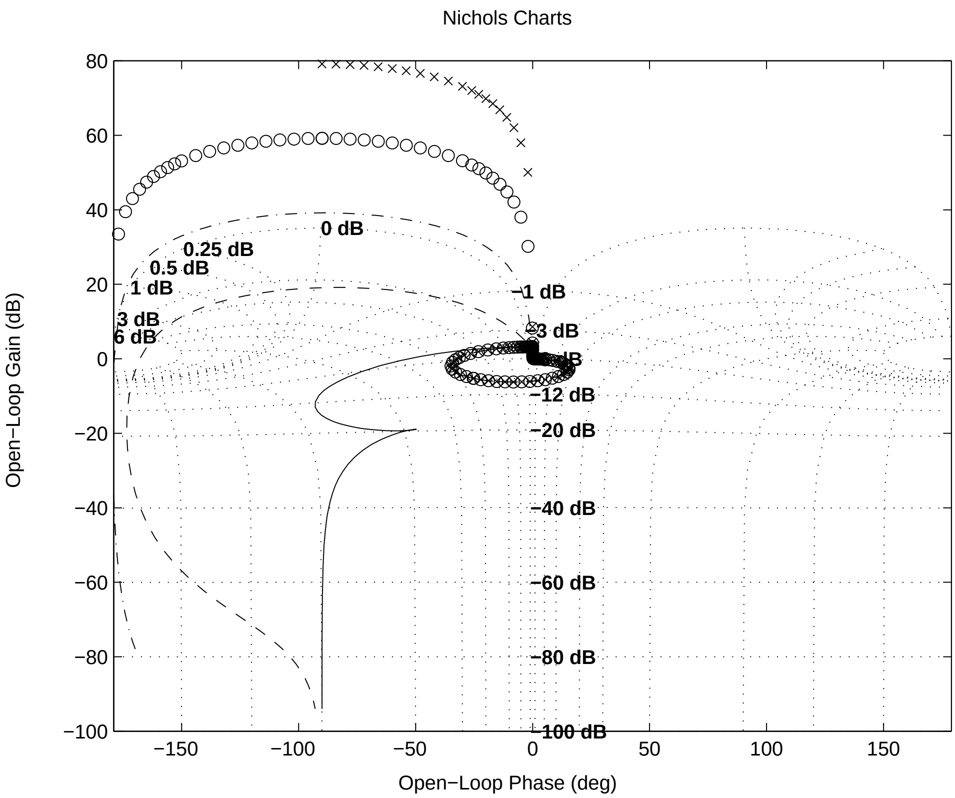


Figure 84: *Nichols plot ($\varepsilon = 1$ as a solid line, $\varepsilon = 0.001$ a dashed line, $\varepsilon = 10^{-5}$ a dashdot line, both $\varepsilon = 10^{-7}$ and $\varepsilon = 10^{-9}$ as dotted lines)*

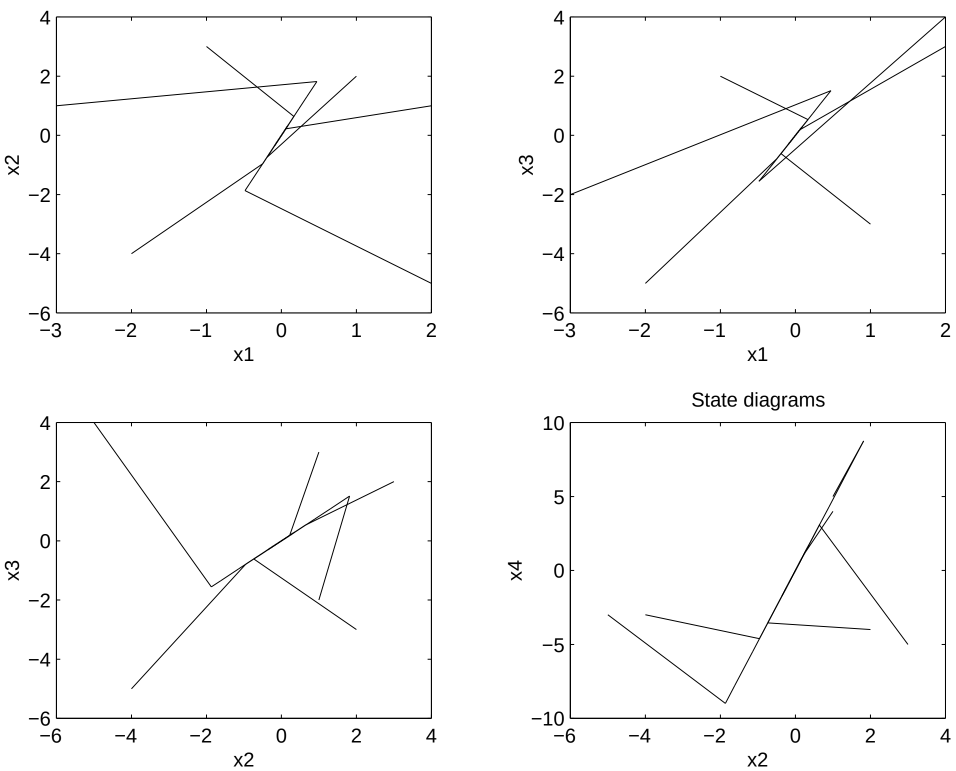


Figure 85: *State diagrams of a system which $\varepsilon = 1$.*

Incorporation of ε

The system equation

$$\begin{aligned} \dot{x} &= E^{-1}Ax + E^{-1}Bu \quad (36) \\ y &= Cx + Du. \quad (37) \end{aligned}$$

where $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon \end{bmatrix}$

Some knowledge about the new system

The equilibrium point still remains at the origin but the eigenvalues are affected by the ε introduced and are

$$-4 \pm \jmath 2, \quad \text{and} \quad -\frac{11}{2} \pm \frac{1}{2}\sqrt{121 - 40\varepsilon}$$

Sample plots of the new system with different ε's

Figure 86 shows a state diagram of a system which is equivalent to the system of Figure 85 but has a different set of initial conditions.

Discussion

The two poles which are not affected by the value of ε (the ones located at $-4 \pm 2\imath$.) have the natural frequency $\omega_n = 4.4721$ and the damping factor 0.8944. The other two poles have the natural frequencies (ω_n) and the damping factors (ζ) as shown in Figure 89 and Figure 90 respectively.

From Figure 72 to Figure 76 one can see that the two zeros at $-2 \pm 5\imath$ remain at the same position what ever the value of ϵ may be while the other remain zero moves to the left away from the origin as $\varepsilon \to 0$ at a high rate as shown in Figure 91.

Figure 89 shows that $\omega_n \to \infty$ as $\varepsilon \to 0$ whie Figure 90 shows that $\zeta \to 0$ as $\varepsilon \to 0$.

Both Figure 77 and Figure 78 show that the resonance freuency (ω_r) increases as ε decreases. Figure 72 to Figure 76 show that the system is a minimum phase system $\forall \varepsilon > 0$, Figure 77, Figure 78, Figure 81 and Figure 82 also show that this is true because the phase [Oga70]

$$\phi \to \deg -90(q - p) = \deg -90(4 - 3) = \deg -90, \quad \text{as} \quad \omega \to \infty,$$

and the log-magnitude curve is

$$-20(q - p) = -20(4 - 3) = -20 \ \frac{\text{db}}{\text{decade}} \quad \text{as}\omega \to \infty.$$

Here p and q are the degrees of the numerator and denominator polynomials of the transfer function respectively.

The Nyquist plot in Figure 79 shows that the system is always stable since [Oga70]

$$Z = N = P = 0 \quad \text{satisfies} \quad Z = N + P,$$

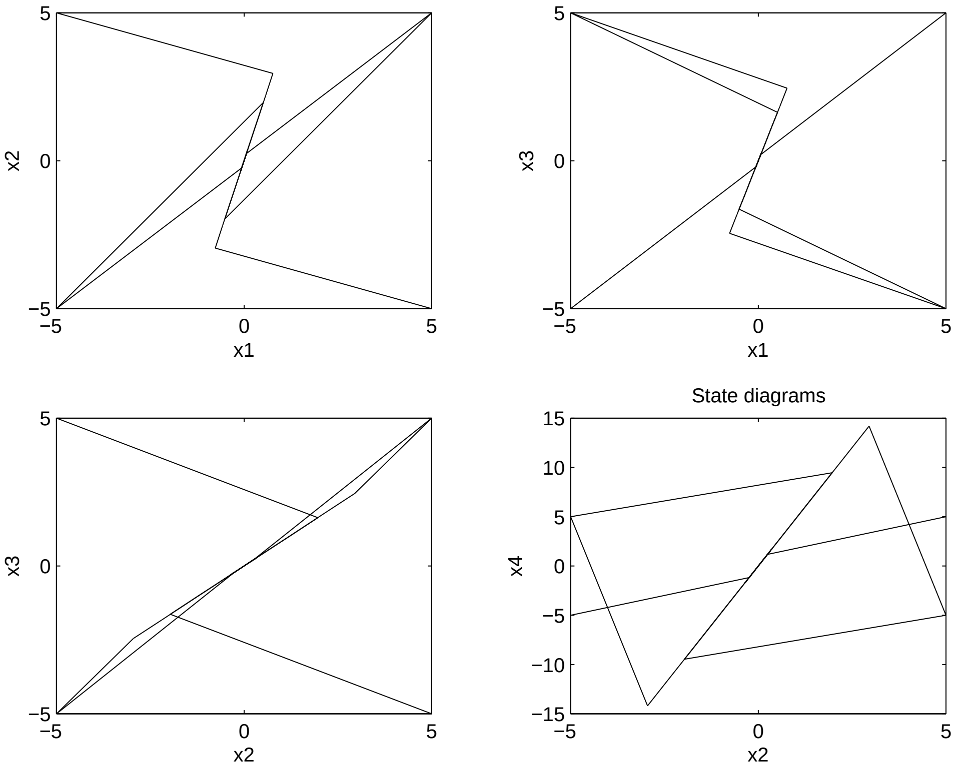


Figure 86: *State diagrams of a system with $\varepsilon = 1$.*

where Z is the number of zeros of $1+G(s)H(s)$ in the right-half s plane, N the number of clockwise enciclements of the $-1+\jmath 0$ point and P the number of poles of $G(S)H(s)$ in the right-half s plane.

Both the Nichols plots of Figure 80 and Figure 84 show that at a given value of gain K the peak or resonance value of M,

$$M_r = \left|\frac{G}{1+G}\right|$$

increases as ε decreases.

With an input variable-structural in nature as shown from Figure 85 to Figure 88 the trajectories switch when coming to a switching hyperplane.

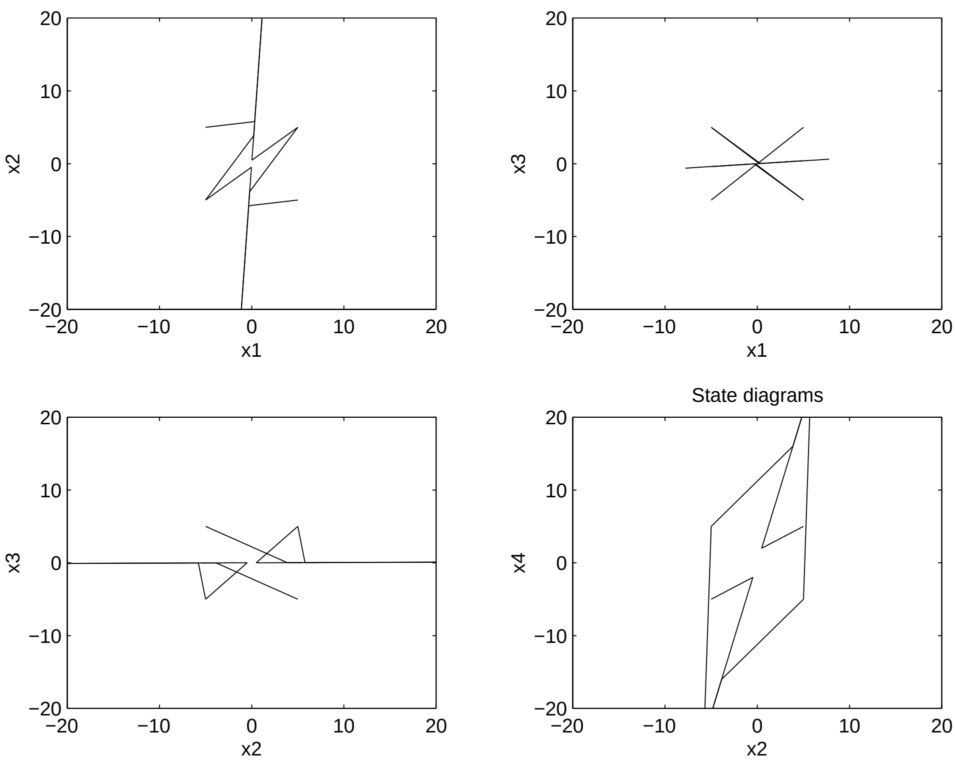


Figure 87: *State diagrams of a system with $\varepsilon = 0.1$.*

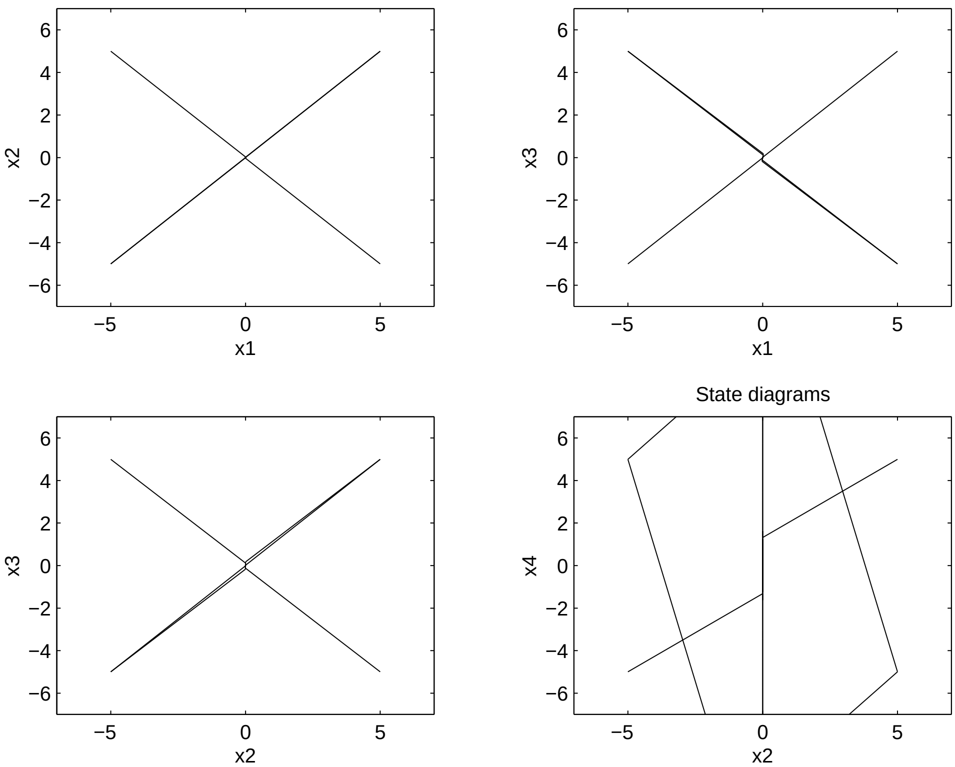


Figure 88: *State diagrams of a system with* $\varepsilon = 0.0001$.

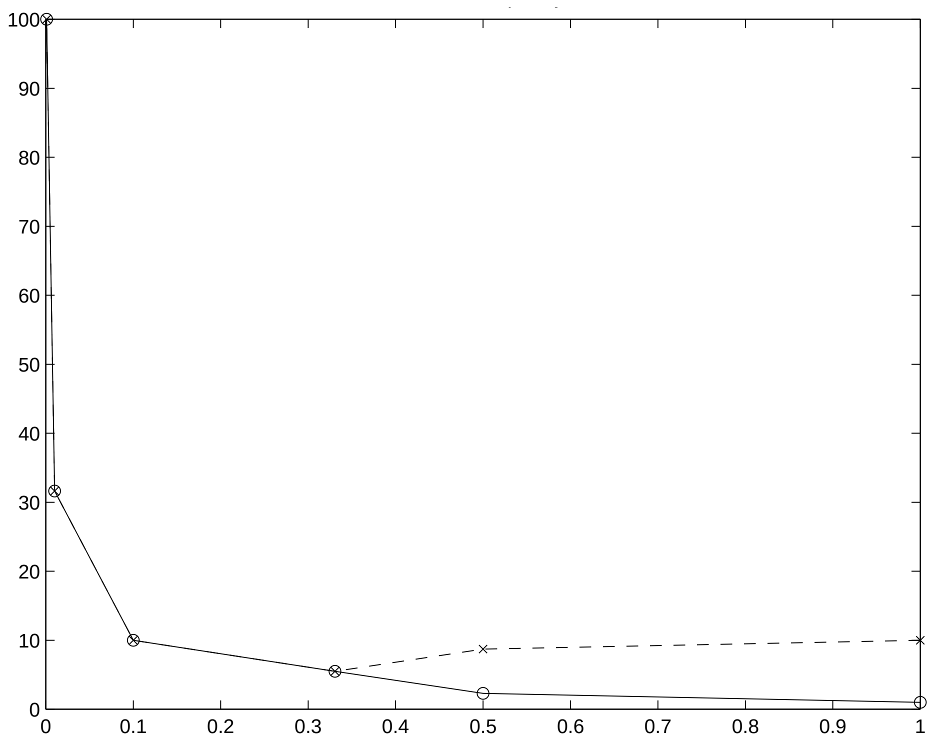


Figure 89: *Natural frequency v.s.* ε

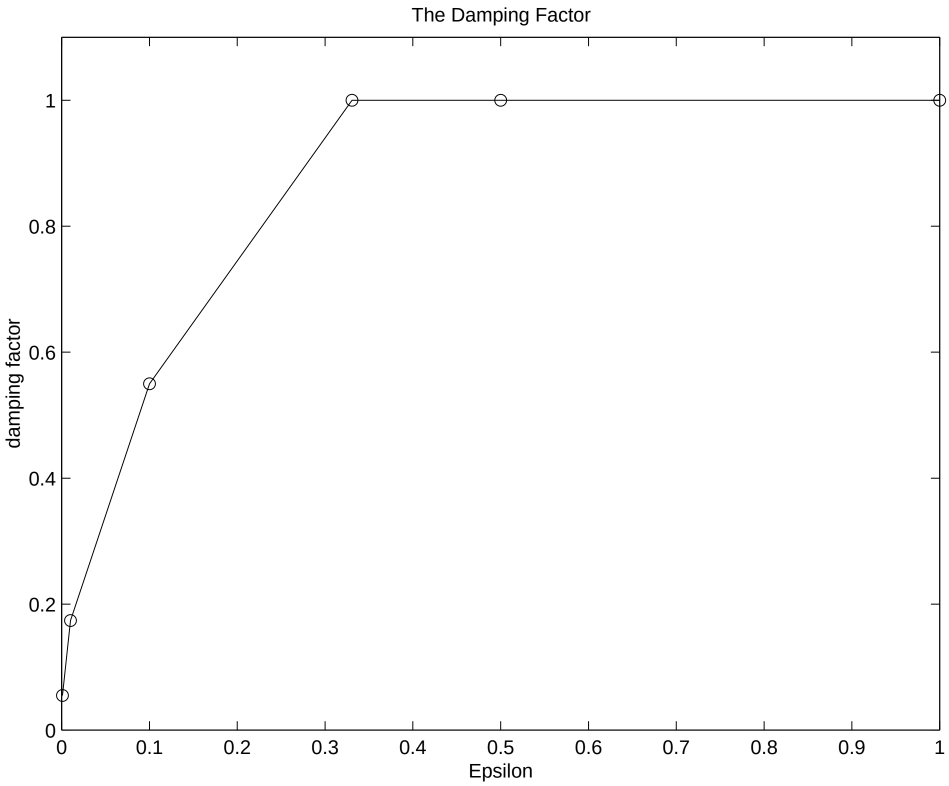


Figure 90: *Damping factor v.s.* ε

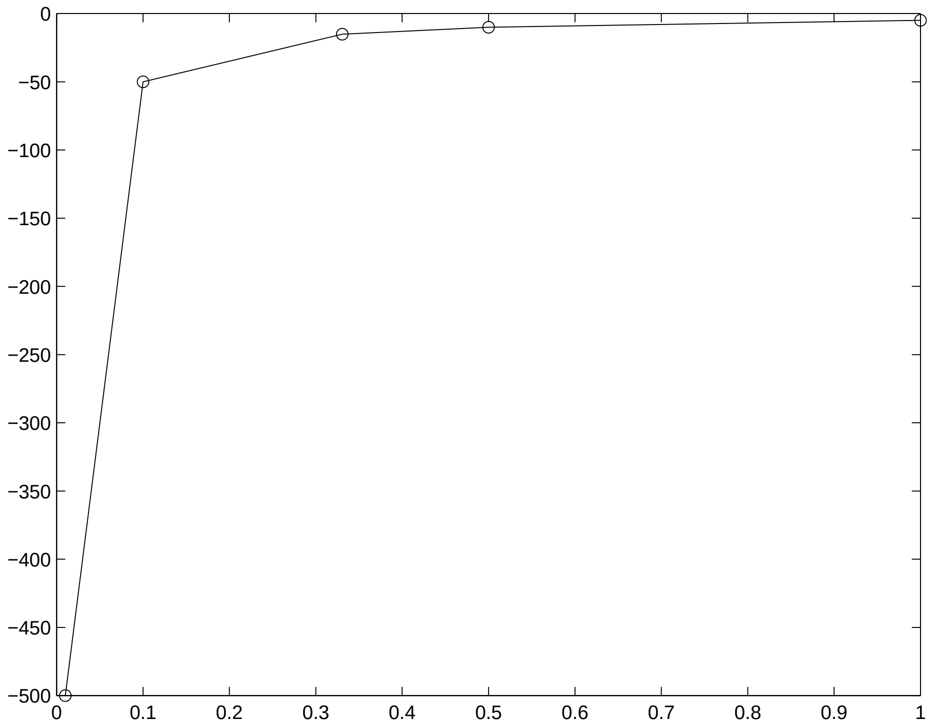


Figure 91: *The position of the moving zero v.s.* ε

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